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What You Should Learn

- Verify that two matrices are inverses of each other.
- Use Gauss-Jordan elimination to find the inverses of matrices.
- Use a formula to find the inverses of 2 × 2 matrices.
- Use inverse matrices to solve systems of linear equations.



The Inverse of a Matrix

The Inverse of a Matrix

Solve ax = b.

To solve this equation for x, multiply each side of the equation by a^{-1} (provided that $a \neq 0$).

$$ax = b$$
$$(a^{-1}a)x = a^{-1}b$$
$$(1)x = a^{-1}b$$
$$x = a^{-1}b$$

The number a^{-1} is called the *multiplicative inverse of a* because $a^{-1}a = 1$.

The Inverse of a Matrix

Definition of the Inverse of a Square Matrix

Let *A* be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

 $AA^{-1} = I_n = A^{-1}A$

then A^{-1} is called the **inverse** of A. The symbol A^{-1} is read "A inverse."

Example 1 – The Inverse of a Matrix

Show that *B* is the inverse of *A*, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Solution:

To show that *B* is the inverse of *A*, show that AB = I = BA, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



cont'd

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As you can see, AB = I = BA.



- If a matrix *A* has an inverse, *A* is called **invertible** (or **nonsingular**);
- otherwise,
- A is called **singular**.
- A nonsquare matrix cannot have an inverse.

Example 2 – Finding the Inverse of a Matrix

Find the inverse of
$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$
.

Solution:

To find the inverse of A, try to solve the matrix equation AX = I for x.

$$\begin{array}{c|cccc} A & X & I \\ & \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \hline x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2 – Solution

cont'd

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases}$$

Linear system with two variables, x_{11} and x_{21} .

$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

Linear system with two variables, x_{12} and x_{22} .

Solve the first system using elementary row operations to determine that $x_{11} = -3$ and $x_{21} = 1$.

From the second system you can determine that $x_{12} = -4$ and $x_{22} = 1$.



cont'd

Therefore, the inverse of A is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Check:

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$
$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 4 & \vdots & 1 \\ -1 & -3 & \vdots & 0 \end{bmatrix}$$

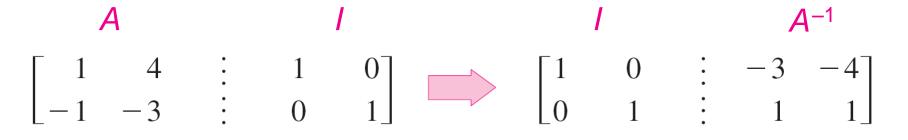
and

 $\begin{bmatrix} 1 & 4 & \vdots & 0 \\ -1 & -3 & \vdots & 1 \end{bmatrix}$

Separately, you can solve them *simultaneously* by *adjoining* the identity matrix to the coefficient matrix to obtain

This "doubly augmented" matrix can be represented as $[A \\ \vdots I]$.

$$\begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{bmatrix}$$
$$R_{1} + R_{2} \rightarrow \begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix}$$
$$-4R_{2} + R_{1} \rightarrow \begin{bmatrix} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix}$$



Finding an Inverse Matrix

Let A be a square matrix of order n.

- **1.** Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \\ \vdots I]$.
- 2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \\ \vdots I]$. The result will be the matrix $[I \\ \vdots A^{-1}]$. If this is not possible, A is not invertible.
- **3.** Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.



The Inverse of a 2 × 2 Matrix

The Inverse of a 2 × 2 Matrix

This simple formula, which works *only* for 2×2 matrices, is explained as follows. If *A* is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
 Formula for inverse of matrix A

The denominator ad - bc is called the **determinant** of the 2×2 matrix A.

Example 4 – Finding the Inverse of a 2 × 2 Matrix

If possible, find the inverse of each matrix.

a.
$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$
 b. $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Solution:

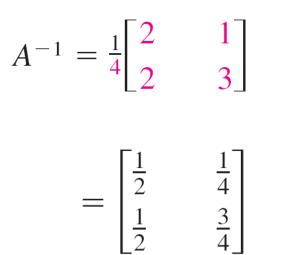
a. For the matrix A, apply the formula for the inverse of a 2×2 matrix to obtain

$$ad - bc = (3)(2) - (-1)(-2)$$

= 4.

Example 4 – Solution

cont'd



Substitute for *a*, *b*, *c*, *d*, and the determinant.

Multiply by the scalar $\frac{1}{4}$.

b. For the matrix *B*, you have

$$ad - bc = (3)(2) - (-1)(-6)$$

= 0

which means that *B* is not invertible.



Systems of Linear Equations

Systems of Linear Equations

A System of Equations with a Unique Solution

If *A* is an invertible matrix, the system of linear equations represented by AX = B has a unique solution given by

 $X = A^{-1}B.$

Example 5 – Solving a System Using an Inverse Matrix

You are going to invest \$10,000 in AAA-rated bonds, AA-rated bonds, and B-rated bonds and want an annual return of \$730. The average yields are 6% on AAA bonds, 7.5% on AA bonds, and 9.5% on B bonds. You will invest twice as much in AAA bonds as in B bonds. Your investment can be represented as

$$\begin{cases} x + y + z = 10,000 \\ 0.06x + 0.075y + 0.095z = 730 \\ x - 2z = 0 \end{cases}$$

where *x*, *y*, and *z* represent the amounts invested in AAA, AA, and B bonds, respectively. Use an inverse matrix to solve the system.

Example 5 – Solution

Begin by writing the system in the matrix form AX = B.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.075 & 0.095 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10,000 \\ 730 \\ 0 \end{bmatrix}$$

Then, use Gauss-Jordan elimination to find A^{-1} .

$$A^{-1} = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix}$$

cont'd

Finally, multiply *B* by A^{-1} on the left to obtain the solution.

$$X = A^{-1}B = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix} \begin{bmatrix} 10,000 \\ 730 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 4000 \\ 4000 \\ 2000 \end{bmatrix}$$

The solution of the system is x = 4000, y = 4000, and z = 2000. So, you will invest \$4000 in AAA bonds, \$4000 in AA bonds, and \$2000 in B bonds.