Linear Programming: Chapter 5
Duality

Robert J. Vanderbei

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Operations Research and Financial Engineering
Princeton University
Princeton, NJ 08544
http://www.princeton.edu/∼rvdb

Resource Allocation

Recall the resource allocation problem ($m = 2$, $n = 3$):

$$\begin{array}{lrcrcrcl} \text{maximize} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\ \text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & \leq & b_1 \\ & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & \leq & b_2 \\ & & & & x_1, & x_2, \; x_3 & \geq & 0, \end{array}$$

where

$$\begin{array}{rcl} c_j & = & \text{profit per unit of product } j \text{ produced} \\ b_i & = & \text{units of raw material } i \text{ on hand} \\ a_{ij} & = & \text{units raw material } i \text{ required to produce } 1 \text{ unit of prod } j. \end{array}$$

Closing Up Shop

If we produce one unit less of product j, then we free up:

- a_{1j} units of raw material 1 and
- a_{2j} units of raw material 2.

Selling these unused raw materials for y_1 and y_2 dollars/unit yields $a_{1j}y_1 + a_{2j}y_2$ dollars.
Only interested if this exceeds lost profit on each product j:

$$a_{1j}y_1 + a_{2j}y_2 \geq c_j, \qquad j = 1, 2, 3.$$

Consider a buyer offering to purchase our entire inventory.
Subject to above constraints, buyer wants to minimize cost:

$$\begin{array}{lrcrcl} \text{minimize} & b_1y_1 & + & b_2y_2 & & \\ \text{subject to} & a_{11}y_1 & + & a_{21}y_2 & \geq & c_1 \\ & a_{12}y_1 & + & a_{22}y_2 & \geq & c_2 \\ & a_{13}y_1 & + & a_{23}y_2 & \geq & c_3 \\ & & & y_1, \; y_2 & \geq & 0 \end{array} .$$

Duality

Every Problem:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 1, 2, \ldots, n, \end{array}$$

Has a Dual:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{m} b_i y_i & \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} y_i a_{ij} \geq c_j & j = 1, 2, \ldots, n \\ & y_i \geq 0 & i = 1, 2, \ldots, m. \end{array}$$

Dual of Dual

Primal Problem:

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & i = 1, \ldots, m \\ & x_j \ge 0 & j = 1, \ldots, n, \end{array}$$

Dual in "Standard" Form:

$$\begin{array}{lll} -\text{maximize} & \sum_{i=1}^{m} -b_i y_i & \\ \text{subject to} & \sum_{i=1}^{m} -a_{ij} y_i \le -c_j & j = 1, \ldots, n \\ & y_i \ge 0 & i = 1, \ldots, m. \end{array}$$

Original problem is called the *primal problem*.

A problem is defined by its data (notation used for the variables is arbitrary).

Dual is "negative transpose" of primal.

Theorem *Dual of dual is primal.*

Weak Duality Theorem

If $(x_1, x_2, \ldots, x_n)$ *is feasible for the primal and* $(y_1, y_2, \ldots, y_m)$ *is feasible for the dual, then*

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

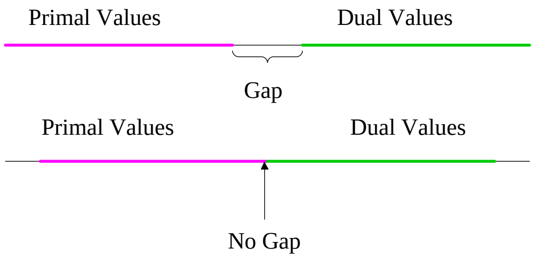
Proof.

$$\begin{aligned}
\sum_j c_j x_j &\leq \sum_j \left(\sum_i y_i a_{ij} \right) x_j \\
&= \sum_{ij} y_i a_{ij} x_j \\
&= \sum_i \left(\sum_j a_{ij} x_j \right) y_i \\
&\leq \sum_i b_i y_i.
\end{aligned}$$

Gap or No Gap?

An important question:

> Is there a gap between the largest primal value and the smallest dual value?



Answer is provided by the Strong Duality Theorem (coming later).

Simplex Method and Duality

A Primal Problem:

obj	=	0	+	−3	x1	+	2	x2	+	1 x3
w1	=	0	−	0	x1	−	−1	x2	−	2 x3
w2	=	3	−	−3	x1	−	4	x2	−	1 x3

Its Dual:

obj	=	0	+	0	y1	+	−3	y2
z1	=	3	−	0	y1	−	3	y2
z2	=	−2	−	1	y1	−	−4	y2
z3	=	−1	−	−2	y1	−	−1	y2

Notes:

- Dual is negative transpose of primal.
- Primal is feasible, dual is not.

Use primal to choose pivot: x_2 enters, w_2 leaves.
Make analogous pivot in dual: z_2 leaves, y_2 enters.

Second Iteration

After First Pivot:

Primal (feasible):

obj	=	3/2	+	-3/2	x1	+	-1/2	w2	+	1/2	x3
w1	=	3/4	−	-3/4	x1	−	1/4	w2	−	9/4	x3
x2	=	3/4	−	-3/4	x1	−	1/4	w2	−	1/4	x3

Dual (still not feasible):

obj	=	-3/2	+	-3/4	y1	+	-3/4	z2
z1	=	3/2	−	3/4	y1	−	3/4	z2
y2	=	1/2	−	-1/4	y1	−	-1/4	z2
z3	=	-1/2	−	-9/4	y1	−	-1/4	z2

Note: *negative transpose property intact.*

Again, use primal to pick pivot: x_3 enters, w_1 leaves.

Make analogous pivot in dual: z_3 leaves, y_1 enters.

After Second Iteration

Primal:

obj	=	5/3	+	-4/3	x1	+	-5/9	w2	+	-2/9 w1
x3	=	1/3	-	-1/3	x1	-	1/9	w2	-	4/9 w1
x2	=	2/3	-	-2/3	x1	-	2/9	w2	-	-1/9 w1

- Is *optimal*.

Dual:

obj	=	-5/3	+	-1/3	z3	+	-2/3 z2
z1	=	4/3	-	1/3	z3	-	2/3 z2
y2	=	5/9	-	-1/9	z3	-	-2/9 z2
y1	=	2/9	-	-4/9	z3	-	1/9 z2

- Negative transpose property remains intact.
- Is *optimal*.

Conclusion

Simplex method applied to primal problem (two phases, if necessary), solves both the primal and the dual.

Strong Duality Theorem

Conclusion on previous slide is the essence of the *strong duality theorem* which we now state:

Theorem. *If the primal problem has an optimal solution,*

$$x^* = (x_1^*, x_2^*, \ldots, x_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \ldots, y_m^*),$$

and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

Paraphrase:

If primal has an optimal solution, then there is no duality gap.

Duality Gap

Four possibilities:

- Primal optimal, dual optimal (no gap).
- Primal unbounded, dual infeasible (no gap).
- Primal infeasible, dual unbounded (no gap).
- Primal infeasible, dual infeasible (infinite gap).

Example of *infinite gap*:

$$\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & x_1 & - & x_2 & \leq & 1 \\
& -x_1 & + & x_2 & \leq & -2 \\
& & x_1, & x_2 & \geq & 0.
\end{array}$$

Complementary Slackness

Theorem. *At optimality, we have*

$$\begin{aligned} x_j z_j &= 0, \qquad \text{for } j = 1, 2, \ldots, n, \\ w_i y_i &= 0, \qquad \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

Proof

Recall the proof of the Weak Duality Theorem:

$$\begin{aligned}
\sum_j c_j x_j \;&\le\; \sum_j (c_j + z_j) x_j = \sum_j \left(\sum_i y_i a_{ij} \right) x_j = \sum_{ij} y_i a_{ij} x_j \\
&=\; \sum_i \left(\sum_j a_{ij} x_j \right) y_i = \sum_i (b_i - w_i) y_i \le \sum_i b_i y_i,
\end{aligned}$$

The inequalities come from the fact that

$$\begin{aligned}
x_j z_j \;&\ge\; 0, \quad \text{for all } j, \\
w_i y_i \;&\ge\; 0, \quad \text{for all } i.
\end{aligned}$$

By Strong Duality Theorem, the inequalities are equalities at optimality.

Dual Simplex Method

When: dual feasible, primal infeasible (i.e., pinks on the left, not on top).

An Example. Showing both primal and dual dictionaries:

obj	=	0.0	+	-2.0	x1	+	-4.0	x2	+	0.0	x3	+	-6.0	x4
w1	=	-3.0	−	-1.0	x1	−	2.0	x2	−	0.0	x3	−	-1.0	x4
w2	=	-5.0	−	2.0	x1	−	-3.0	x2	−	0.0	x3	−	-2.0	x4
w3	=	8.0	−	2.0	x1	−	3.0	x2	−	3.0	x3	−	2.0	x4

obj	=	0.0	+	3.0	y1	+	5.0	y2	+	-8.0	y3
z1	=	2.0	−	1.0	y1	−	-2.0	y2	−	-2.0	y3
z2	=	4.0	−	-2.0	y1	−	3.0	y2	−	-3.0	y3
z3	=	0.0	−	0.0	y1	−	0.0	y2	−	-3.0	y3
z4	=	6.0	−	1.0	y1	−	2.0	y2	−	-2.0	y3

Looking at dual dictionary: y_2 enters, z_2 leaves.

On the primal dictionary: w_2 leaves, x_2 enters.

After pivot...

Dual Simplex Method: Second Pivot

Going in, we have:

obj	=	-6.6667	+	-4.6667	x1	+	-1.3333	w2	+	0.0	x3	+	-3.3333	x4
w1	=	-6.3333	-	0.3333	x1	-	0.6667	w2	-	0.0	x3	-	-2.3333	x4
x2	=	1.6667	-	-0.6667	x1	-	-0.3333	w2	-	0.0	x3	-	0.6667	x4
w3	=	3.0	-	4.0	x1	-	1.0	w2	-	3.0	x3	-	0.0	x4

obj	=	6.6667	+	6.3333	y1	+	-1.6667	z2	+	-3.0	y3
z1	=	4.6667	-	-0.3333	y1	-	0.6667	z2	-	-4.0	y3
y2	=	1.3333	-	-0.6667	y1	-	0.3333	z2	-	-1.0	y3
z3	=	0.0	-	0.0	y1	-	0.0	z2	-	-3.0	y3
z4	=	3.3333	-	2.3333	y1	-	-0.6667	z2	-	0.0	y3

Looking at dual: y_1 enters, z_4 leaves.

Looking at primal: w_1 leaves, x_4 enters.

Dual Simplex Method Pivot Rule

obj	=	-6.6667	+	-4.6667	x1	+	-1.3333	w2	+	0.0	x3	+	-3.3333	x4
w1	=	-6.3333	-	0.3333	x1	-	0.6667	w2	-	0.0	x3	-	-2.3333	x4
x2	=	1.6667	-	-0.6667	x1	-	-0.3333	w2	-	0.0	x3	-	0.6667	x4
w3	=	3.0	-	4.0	x1	-	1.0	w2	-	3.0	x3	-	0.0	x4

Refering to the primal dictionary:

- Pick leaving variable from those rows that are *infeasible*.
- Pick entering variable from a box with a negative value and which can be increased the least (on the dual side).

Next primal dictionary shown on next page...

Dual Simplex Method: Third Pivot

Going in, we have:

obj	=	-15.7143	+	-5.1429	x1	+	-2.2857	w2	+	0.0	x3	+	-1.4286	w1
x4	=	2.7143	−	-0.1429	x1	−	-0.2857	w2	−	0.0	x3	−	-0.4286	w1
x2	=	-0.1429	−	-0.5714	x1	−	-0.1429	w2	−	0.0	x3	−	0.2857	w1
w3	=	3.0	−	4.0	x1	−	1.0	w2	−	3.0	x3	−	0.0	w1

Which variable must leave and which must enter?

See next page...

Dual Simplex Method: Third Pivot—Answer

Answer is: x_2 leaves, x_1 enters.

Resulting dictionary is OPTIMAL:

obj	=	-17.0	+	-9.0	x2	+	-1.0	w2	+	0.0	x3	+	-4.0	w1
x4	=	2.75	-	-0.25	x2	-	-0.25	w2	-	0.0	x3	-	-0.5	w1
x1	=	0.25	-	-1.75	x2	-	0.25	w2	-	0.0	x3	-	-0.5	w1
w3	=	2.0	-	7.0	x2	-	0.0	w2	-	3.0	x3	-	2.0	w1

Dual-Based Phase I Method

Example:

obj	=	0.0			+	-4.0	x1	+	2.0	x2	+	3.0	x3
					+	-1.0	x1	+	-1.0	x2	+	-1.0	x3
w1	=	0.0	+	1.0	-	2.0	x1	-	-1.0	x2	-	3.0	x3
w2	=	0.0	+	1.0	-	3.0	x1	-	-3.0	x2	-	-4.0	x3
w3	=	-3.0	+	1.0	-	-1.0	x1	-	-1.0	x2	-	1.0	x3
w4	=	-1.0	+	1.0	-	-2.0	x1	-	0.0	x2	-	0.0	x3

Notes:

- Two objective functions: the true objective (on top), and a fake one (below it).
- For *Phase I*, use the fake objective—it's dual feasible.
- Two right-hand sides: the real one (on the left) and a fake (on the right).
- Ignore the fake right-hand side—we'll use it in another algorithm later.

Phase I—First Pivot: w_3 leaves, x_1 enters.
After first pivot...

Dual-Based Phase I Method—Second Pivot

Recall current dictionary:

obj	=	-12.0			+	-4.0	w3	+	6.0	x2	+	-1.0	x3
					+	-1.0	w3	+	0.0	x2	+	-2.0	x3
w1	=	-6.0	+	3.0	−	2.0	w3	−	-3.0	x2	−	5.0	x3
w2	=	-9.0	+	4.0	−	3.0	w3	−	-6.0	x2	−	-1.0	x3
x1	=	3.0	+	-1.0	−	-1.0	w3	−	1.0	x2	−	-1.0	x3
w4	=	5.0	+	-1.0	−	-2.0	w3	−	2.0	x2	−	-2.0	x3

Dual pivot: w_2 leaves, x_2 enters.

After pivot:

obj	=	-3.0			+	-1.0	w3	+	1.0	w2	+	-2.0	x3
					+	-1.0	w3	+	0.0	w2	+	-2.0	x3
w1	=	-1.5	+	1.0	−	0.5	w3	−	-0.5	w2	−	5.5	x3
x2	=	1.5	+	-0.6667	−	-0.5	w3	−	-0.1667	w2	−	0.1667	x3
x1	=	1.5	+	-0.3333	−	-0.5	w3	−	0.1667	w2	−	-1.1667	x3
w4	=	2.0	+	0.3333	−	-1.0	w3	−	0.3333	w2	−	-2.3333	x3

Dual-Based Phase I Method—Third Pivot

Current dictionary:

obj	=	-3.0			+	-1.0	w3	+	1.0	w2	+	-2.0	x3
					+	-1.0	w3	+	0.0	w2	+	-2.0	x3
w1	=	-1.5	+	1.0	−	0.5	w3	−	-0.5	w2	−	5.5	x3
x2	=	1.5	+	-0.6667	−	-0.5	w3	−	-0.1667	w2	−	0.1667	x3
x1	=	1.5	+	-0.3333	−	-0.5	w3	−	0.1667	w2	−	-1.1667	x3
w4	=	2.0	+	0.3333	−	-1.0	w3	−	0.3333	w2	−	-2.3333	x3

Dual pivot:

w_1 leaves,

w_2 enters.

After pivot:

obj	=	0.0			+	0.0	w3	+	2.0	w1	+	9.0	x3
					+	-1.0	w3	+	0.0	w1	+	-2.0	x3
w2	=	3.0	+	-2.0	−	-1.0	w3	−	-2.0	w1	−	-11.0	x3
x2	=	2.0	+	-1.0	−	-0.6667	w3	−	-0.3333	w1	−	-1.6667	x3
x1	=	1.0	+	0.0	−	-0.3333	w3	−	0.3333	w1	−	0.6667	x3
w4	=	1.0	+	1.0	−	-0.6667	w3	−	0.6667	w1	−	1.3333	x3

It's **feasible!**

Fourth Pivot—Phase II

Current dictionary:

obj	=	0.0			+	0.0	w3	+	2.0	w1	+	9.0	x3
					+	-1.0	w3	+	0.0	w1	+	-2.0	x3
w2	=	3.0	+	-2.0	−	-1.0	w3	−	-2.0	w1	−	-11.0	x3
x2	=	2.0	+	-1.0	−	-0.6667	w3	−	-0.3333	w1	−	-1.6667	x3
x1	=	1.0	+	0.0	−	-0.3333	w3	−	0.3333	w1	−	0.6667	x3
w4	=	1.0	+	1.0	−	-0.6667	w3	−	0.6667	w1	−	1.3333	x3

It's feasible.

Ignore fake objective.

Use the real thing (top row).

Primal pivot: x_3 enters, w_4 leaves.

Final Dictionary

After pivot:

obj	=	6.75			+	4.5	w3	+	-2.5	w1	+	-6.75	w4
					+	-2.0	w3	+	1.0	w1	+	1.5	w4
w2	=	11.25	+	6.25	−	-6.5	w3	−	3.5	w1	−	8.25	w4
x2	=	3.25	+	0.25	−	-1.5	w3	−	0.5	w1	−	1.25	w4
x1	=	0.5	+	-0.5	−	0.0	w3	−	0.0	w1	−	-0.5	w4
x3	=	0.75	+	0.75	−	-0.5	w3	−	0.5	w1	−	0.75	w4

Problem is **unbounded!**