Example (part 1): Simplex method

Solve using the Simplex method the following problem:

Maximize $Z = f(x,y) = 3x + 2y$
subject to: $2x + y \le 18$
$2x + 3y \le 42$
$3x + y \le 24$
$x \ge 0 , y \ge 0$

Consider the following steps:

1. **Make a change of variables and normalize the sign of the independent terms.**

A change is made to the variable naming, establishing the following correspondences:

- x becomes X_1
- y becomes X_2

As the independent terms of all restrictions are positive no further action is required. Otherwise there would be multiplied by "-1" on both sides of the inequality (noting that this operation also affects the type of restriction).

2. **Normalize restrictions.**

The inequalities become equations by adding *slack*, *surplus* and *artificial variables* as the following table:

Inequality type	Variable that appears
≥	- surplus + artificial
=	+ artificial
≤	+ slack

In this case, a slack variable (X_3, X_4 and X_5) is introduced in each of the restrictions of ≤ type, to convert them into equalities, resulting the system of linear equations:

$$2 \cdot X_1 + X_2 + X_3 = 18$$
$$2 \cdot X_1 + 3 \cdot X_2 + X_4 = 42$$
$$3 \cdot X_1 + X_2 + X_5 = 24$$

3. **Match the objective function to zero.**

$$Z - 3 \cdot X_1 - X_2 - 0 \cdot X_3 - 0 \cdot X_4 - 0 \cdot X_5 = 0$$

4. **Write the initial tableau of Simplex method.**

The initial tableau of Simplex method consists of all the coefficients of the decision variables of the original problem and the slack, surplus and artificial variables added in second step (in columns, with P_0 as the constant term and P_i as the coefficients of the rest of X_i variables), and constraints (in rows). The C_b column contains the coefficients of the variables that are in the base.

The first row consists of the objective function coefficients, while the last row contains the objective function value and *reduced costs* $Z_j - C_j$.

The last row is calculated as follows: $Z_j = \Sigma(C_{bi} \cdot P_j)$ for i = 1..m, where if j = 0, $P_0 = b_i$ and $C_0 = 0$, else $P_j = a_{ij}$. Although this is the first tableau of the Simplex method and all C_b are null, so the calculation can simplified, and by this time $Z_j = -C_j$.

Tableau I . 1st iteration							
			3	2	0	0	0
Base	**C_b**	**P_0**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
P_3	0	18	2	1	1	0	0
P_4	0	42	2	3	0	1	0
P_5	0	24	**3**	1	0	0	1
Z		0	-3	-2	0	0	0

5. **Stopping condition.**

If the objective is to maximize, when in the last row (indicator row) there is no negative value between discounted costs (P_1 columns below) the stop condition is reached.

In that case, the algorithm reaches the end as there is no improvement possibility. The Z value (P_0 column) is the optimal solution of the problem.

Another possible scenario is all values are negative or zero in the input variable column of the base. This indicates that the problem is not limited and the solution will always be improved.

Otherwise, the following steps are executed iteratively.

6. **Choice of the input and output base variables.**

First, input base variable is determined. For this, column whose value in Z row is the lesser of all the negatives is chosen. In this example it would be the variable X_1 (P_1) with -3 as coefficient.

If there are two or more equal coefficients satisfying the above condition (case of tie), then choice the basic variable.

The column of the input base variable is called *pivot column* (in **green** color).

Once obtained the input base variable, the output base variable is determined. The decision is based on a simple calculation: divide each independent term (P_0 column) between the corresponding value in the pivot column, if both values are strictly positive (greater than zero). The row whose result is minimum score is chosen.

If there is any value less than or equal to zero, this quotient will not be performed. If all values of the pivot column satisfy this condition, the stop condition will be reached and the problem has an unbounded solution (see Simplex method theory).

In this example: 18/2 [=9] , 42/2 [=21] and 24/3 [=8]

The term of the pivot column which led to the lesser positive quotient in the previous division indicates the row of the slack variable leaving the base. In this example, it is X_5 (P_5), with 3 as coefficient. This row is called *pivot row* (in **green**).

If two or more quotients meet the choosing condition (case of tie), other than that basic variable is chosen (wherever possible).

The intersection of *pivot column* and *pivot row* marks the *pivot value*, in this example, 3.

7. **Update tableau.**

The new coefficients of the tableau are calculated as follows:

- In the pivot row each new value is calculated as:
  New value = Previous value / Pivot
- In the other rows each new value is calculated as:
  *New value = Previous value - (Previous value in pivot column * New value in pivot row)*

So the pivot is normalized (its value becomes 1), while the other values of the pivot column are canceled (analogous to the Gauss-Jordan method).

Calculations for P_4 row are shown below:

Previous P_4 row	42	2	3	0	1	0
	-	-	-	-	-	-
Previous value in pivot column	2	2	2	2	2	2
	x	x	x	x	x	x
New value in pivot row	8	1	1/3	0	0	1/3

	=	=	=	=	=	=
New P_4 row	26	0	7/3	0	1	-2/3

The tableau corresponding to this second iteration is:

Tableau II . 2nd iteration

			3	2	0	0	0
Base	**C_b**	**P_0**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
P_3	0	2	0	**1/3**	1	0	-2/3
P_4	0	26	0	7/3	0	1	-2/3
P_1	3	8	1	1/3	0	0	1/3
Z		24	0	-1	0	0	1

8. When checking the stop condition is observed which is not fulfilled since there is one negative value in the last row, -1. So, continue iteration steps 6 and 7 again.

- 6.1. The input base variable is X_2 (P_2), since it is the variable that corresponds to the column where the coefficient is -1.
- 6.2. To calculate the output base variable, the constant terms P_0 column) are divided by the terms of the new pivot column: 2 / 1/3 [=6] , 26 / 7/3 [=78/7] and 8 / 1/3 [=24]. As the lesser positive quotient is 6, the output base variable is X_3 (P_3).
- 6.3. The new pivot is 1/3.
- 7. Updating the values of tableau again is obtained:

Tableau III . 3rd iteration

			3	2	0	0	0
Base	**C_b**	**P_0**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
P_2	2	6	0	1	3	0	-2
P_4	0	12	[illegible]	0	-7	1	**4**
P_1	3	6	[illegible]	0	-1	0	1
Z		30	[illegible]	0	3	0	-1

9. Checking again the stop condition reveals that the pivot row has one [illegible]ve value, -1. It means that optimal solution is not reached yet and we must continue iterating (steps 6 and 7):

- 6.1. The input base variable is X_5 (P_5), since it is the variable that corresponds to the column where the coefficient is -1.
- 6.2. To calculate the output base variable, the constant terms (P_0) are divided by the terms of the new pivot column: 6/(-2) [=-3] , 12/4 [=3] , and 6/1 [=6]. In this iteration, the output base variable is X_4 (P_4).
- 6.3. The new pivot is 4.
- 7. Updating the values of tableau again is obtained:

Tableau IV . 4th iteration

			3	2	0	0	0
Base	**C_b**	**P_0**	**P_1**	**P_2**	**P_3**	**P_4**	**P_5**
P_2	2	**12**	0	1	-1/2	1/2	0
P_5	0	3	0	0	-7/4	1/4	1
P_1	3	**3**	1	0	3/4	-1/4	0
Z		**33**	0	0	5/4	1/4	0

10. **End of algorithm.**

It is noted that in the last row, all the coefficients are positive, so the stop condition is fulfilled.

The optimal solution is given by the val-ue of Z in the constant terms column (P_0 column), in the example: 33. In the same column, the point where it reaches is shown, watching the corresponding rows of input decision variables: $X_1 = 3$ and $X_2 = 12$.

Undoing the name change gives x = 3 and y = 12.

Solve with PHPSimplex.