LU decomposition

30.3

Introduction

In this section we consider another direct method for obtaining the solution of systems of equations in the form $AX = B$.

Prerequisites

Before starting this Section you should . . .

① revise matrices and their use in systems of equations

② revise determinants

Learning Outcomes

After completing this Section you should be able to . . .

✓ find an LU decomposition of simple matrices and apply it to solve systems of equations

✓ be aware of when an LU decomposition is unavailable and when it is possible to circumvent the problem

1. LU decomposition

Suppose we have the system of equations

$$AX = B.$$

The motivation for an LU decomposition is based on the observation that systems of equations involving triangular coefficient matrices are easier to deal with. Indeed, the whole point of Gaussian Elimination is to replace the coefficient matrix with one that is triangular. The LU decomposition is another approach designed to exploit triangular systems.
We suppose that we can write

$$A = LU$$

where L is a lower triangular matrix and U is an upper triangular matrix. Our aim is to find L and U and once we have done so we have found an LU decomposition of A.

Key Point

An LU decomposition of a matrix A is the product of a lower triangular matrix and an upper triangular matrix that is equal to A.

It turns out that we need only consider lower triangular matrices L that have 1s down the diagonal. Here is an example, let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$.
Multiplying out LU and setting the answer equal to A gives

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

Now we have to use this to find the entries in L and U. Fortunately this is not nearly as hard as it might at first seem. We begin by running along the top row to see that

$$\boxed{U_{11} = 1}\,, \qquad \boxed{U_{12} = 2}\,, \qquad \boxed{U_{13} = 4}\,.$$

Now consider the second row

$$L_{21}U_{11} = 3 \quad \therefore L_{21} \times 1 = 3 \quad \therefore \boxed{L_{21} = 3}\,,$$

$$L_{21}U_{12} + U_{22} = 8 \quad \therefore 3 \times 2 + U_{22} = 8 \quad \therefore \boxed{U_{22} = 2}\,,$$

$$L_{21}U_{13} + U_{23} = 14 \quad \therefore 3 \times 4 + U_{23} = 14 \quad \therefore \boxed{U_{23} = 2}\,.$$

Notice how, at each step, the equation in hand has only one unknown in it, and other quantities that we have already found. This pattern continues on the last row

$$L_{31}U_{11} = 2 \quad \therefore L_{31} \times 1 = 2 \quad \therefore \boxed{L_{31} = 2}\,,$$

$$L_{31}U_{12} + L_{32}U_{22} = 6 \quad \therefore 2 \times 2 + L_{32} \times 2 = 6 \quad \therefore \boxed{L_{32} = 1}\,,$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13 \quad \therefore (2 \times 4) + (1 \times 2) + U_{33} = 13 \quad \therefore \boxed{U_{33} = 3}\,.$$

We have shown that

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

and this is an LU decomposition of A.

Find an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$.

Your solution

Let

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} \end{bmatrix}$$

then, comparing the left and right hand sides row by row implies that $U_{11} = 3$, $U_{12} = 1$, $L_{21}U_{11} = -6$ which implies $L_{21} = -2$ and $L_{21}U_{12} + U_{22} = -4$ which implies that $U_{22} = -2$.
Hence

$$\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$$

is an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$.

Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

Your solution

Using material from the worked example in the notes we set

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

and comparing elements row by row we see that

$$\begin{array}{lll} U_{11} = 3, & U_{12} = 1, & U_{13} = 6, \\ L_{21} = -2, & U_{22} = 2, & U_{23} = -4 \\ L_{31} = 0 & L_{32} = 4 & U_{33} = -1 \end{array}$$

and it follows that

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

is an LU decomposition of the given matrix.

2. Using an LU decomposition to solve systems of equations

Once a matrix A has been decomposed into lower and upper triangular parts it is possible to obtain the solution to $AX = B$ in a direct way.
The procedure can be summarised as follows

- Given A, find L and U so that $A = LU$. Hence $LUX = B$.
- Let $Y = UX$ so that $LY = B$. Solve this triangular system for Y.
- Finally solve the triangular system $UX = Y$ for X.

The benefit of this approach is that we only ever need to solve triangular systems. The cost is that we have to solve two of them.

Example Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

Solution

- The first step is to calculate the LU decomposition of the coefficient matrix on the left-hand side. In this case that job has already been done since this is the matrix we considered earlier. We found that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

- The next step is to solve $LY = B$ for the vector $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. That is we consider

$$LY = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} = B$$

which can be solved by **forward substitution**. From the top equation we see that $y_1 = 3$. The middle equation states that $3y_1 + y_2 = 13$ and hence $y_2 = 4$. Finally the bottom line says that $2y_1 + y_2 + y_3 = 4$ from which we see that $y_3 = -6$.

Solution (contd.)

- Now that we have found Y we finish the procedure by solving $UX = Y$ for X. That is we solve

$$UX = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} = Y$$

  by using **back substitution**. Starting with the bottom equation we see that $3x_3 = -6$ so clearly $x_3 = -2$. The middle equation implies that $2x_2 + 2x_3 = 4$ and it follows that $x_2 = 4$. The top equation states that $x_1 + 2x_2 + 4x_3 = 3$ and consequently $x_1 = 3$.

Therefore we have found that the solution to the system of simultaneous equations

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \qquad \text{is} \qquad X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

Use the LU decomposition you found earlier in this Section to solve
$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}.$$

Your solution

We found earlier that the coefficient matrix is equal to $LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$.

First we solve $LY = B$ for Y, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}.$$

The top line implies that $y_1 = 0$. The middle line states that $-2y_1 + y_2 = 4$ and therefore $y_2 = 4$. The last line tells us that $4y_2 + y_3 = 17$ and therefore $y_3 = 1$. Finally we solve $UX = Y$ for X, we have

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

The bottom line shows that $x_3 = -1$. The middle line then shows that $x_2 = 0$, and then the top line gives us that $x_1 = 2$. The required solution is $X = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

3. Do matrices always have an LU decomposition?

No. Sometimes it is impossible to write a matrix in the form

$$\text{“lower triangular”} \times \text{“upper triangular”}.$$

Why not?

An invertible matrix A has an LU decomposition provided that all its **leading submatrices** have non-zero determinants. The k-th leading submatrix of A is denoted A_k and is the $k \times k$ matrix found by looking only at the top k rows and leftmost k columns. For example if

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

then the leading submatrices are

$$A_1 = 1, \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

The fact that this matrix A has an LU decomposition can be guaranteed in advance because none of these determinants is zero:

$$\begin{aligned}
|A_1| &= 1, \\
|A_2| &= (1 \times 8) - (2 \times 3) = 2, \\
|A_3| &= \begin{vmatrix} 8 & 14 \\ 6 & 13 \end{vmatrix} - 2\begin{vmatrix} 3 & 14 \\ 2 & 13 \end{vmatrix} + 4\begin{vmatrix} 3 & 8 \\ 2 & 6 \end{vmatrix} = 20 - (2 \times 11) + (4 \times 2) = 6
\end{aligned}$$

(where the 3×3 determinant was found by expanding along the top row).

Example Show that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ does not have an LU decomposition.

Solution

The second leading submatrix has determinant equal to

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 2) = 0$$

which means that an LU decomposition is not possible in this case.

Which, if any, of these matrices have an LU decomposition?

(i) $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, (ii) $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, (iii) $A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$.

Your solution

(i)

$|A_1| = 3$ and $|A_2| = |A| = 3$. Neither of these is zero, so A does have an LU decomposition.

Your solution

(ii)

$|A_1| = 0$ so A does not have an LU decomposition.

Your solution

(iii)

$|A_1| = 1$, $|A_2| = 6 - 6 = 0$, so A does not have an LU decomposition.

The example below gives some strong evidence for the key result being stated in this section.

Can we get around this problem?

Yes. It is always possible to re-order the rows of an invertible matrix so that all of the submatrices have non-zero determinants.

Example Reorder the rows of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$ so that the reordered matrix has an LU decomposition.

Solution

Swapping the first and second rows doesn't help us since the second leading submatrix will still have a zero determinant. Let us swap the second and third rows and consider

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

the leading submatrices are

$$B_1 = 1, \quad B_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B_3 = B.$$

Now $|B_1| = 1$, $|B_2| = 3 \times 1 - 2 \times 1 = 1$ and (expanding along the first row)

$$|B_3| = 1(15 - 16) - 2(5 - 8) + 3(4 - 6) = -1 + 6 - 6 = -1.$$

All three of these determinants are non-zero and we conclude that B does indeed have an LU decomposition.

Reorder the rows of $A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$ so that the reordered matrix has an LU decomposition.

Your solution

Let us swap the second and third rows and consider

$$B = \begin{bmatrix} 1 & -3 & 7 \\ 0 & 3 & -2 \\ -2 & 6 & 1 \end{bmatrix}$$

the leading submatrices are

$$B_1 = 1, \quad B_2 = \begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix}, \quad B_3 = B$$

which have determinants 1, 3 and 45 respectively. All of these are non-zero and we conclude that B does indeed have an LU decomposition.

Exercises

1. Calculate LU decompositions for each of these matrices

   (a) $A = \begin{bmatrix} 2 & 1 \\ -4 & -6 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & 1 & -4 \\ 2 & 1 & -2 \\ 6 & 3 & -11 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 5 \\ 1 & 11 & 4 \end{bmatrix}$

2. Check each answer in Question 1, by multiplying out LU to show that the product is equal to A.

3. Using the answers obtained in Question 1, solve the following systems of equations.

   (a) $\begin{bmatrix} 2 & 1 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 1 & -4 \\ 2 & 1 & -2 \\ 6 & 3 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \\ 41 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 5 \\ 1 & 11 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

4. Consider $A = \begin{bmatrix} 1 & 6 & 2 \\ 2 & 12 & 5 \\ -1 & -3 & -1 \end{bmatrix}$

   (a) Show that A does not have an LU decomposition.

   (b) Re-order the rows of A and find an LU decomposition of the new matrix.

   (c) Hence solve

$$\begin{aligned} x_1 + 6x_2 + 2x_3 &= 9 \\ 2x_1 + 12x_2 + 5x_3 &= -4 \\ -x_1 - 3x_2 - x_3 &= 17 \end{aligned}$$