LU Decomposition

One way of solving a system of equations is using the Gauss-Jordan method. Another way of solving a system of equations is by using a factorization technique for matrices called LU decompostion. This factorization is involves two matrices, one lower triangular matrix and one upper triangular matrix. First we explain how to find an LU decomposition.

*Hints when finding an LU Decompostion:
1. Row swapping is not allowed. If you swap rows, then an LU decomposition will not exist.
2. It is not necessary to get leading ones on the main diagonal when using Gaussian Elimination. In some matrices, however, it is recommended to get leading ones to use nice row operations.
3. When using Gaussian Elimination to find such an LU decomposition, record all row operations involved. The row operations will help find the lower triangular matrix using the identity matrix.
4. An LU decomposition is not unique. There can be more than one such LU decomposition for a matrix.

To get the matrix U, just use row operations until an upper triangular matrix is formed.

To get L, start with the idenity matrix and use the following rules.

- Any row operations that involves adding a multiple of one row to another, for example, $R_i + \boldsymbol{k}R_j$, put the value $-\boldsymbol{k}$ in the i^{th}-row, j^{th}-column of the identity matrix.
- Any row operations that involves getting a leading one on the main diagonal, for example, $\boldsymbol{k}R_i$, put the value $1/\boldsymbol{k}$ in the position of the identity matrix where the leading one occurs.

Example: Find an LU decomposition of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix}$$

1. Use Gaussian Elimination to get the upper triangular matrix U.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \xrightarrow[R3-3R1]{R2-2R1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & 0 \\ 0 & -15 & -12 \end{bmatrix} \xrightarrow{\frac{-1}{8}R2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -15 & -12 \end{bmatrix} \xrightarrow{R3+15R2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -12 \end{bmatrix} \xrightarrow{\frac{-1}{12}R3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Form the lower triangular matrix L by using the rules mentioned above for the row operations involved to get U.

→ Start with the identity matrix. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ Row operation 1: R2 – 2R1 $\begin{bmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ Row operation 2: R3 – 3R1 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \boxed{3} & 0 & 1 \end{bmatrix}$

→ Row operation 3: –1/8R1 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & \boxed{-8} & 0 \\ 3 & 0 & 1 \end{bmatrix}$

→ Row operation 4: R3 +15R2 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & \boxed{-15} & 1 \end{bmatrix}$

→ Row operation 5: –1/6R3 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & \boxed{-12} \end{bmatrix}$

Thus an LU Decomposition is given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next we show how an LU decomposition can be used to solve a system of equations.

Steps to solve a system using an LU decomposition:
1. Set up the equation A$\mathbf{x}$ = $\mathbf{b}$.
2. Find an LU decomposition for A. This will yield the equation
   (LU)$\mathbf{x}$ = $\mathbf{b}$.
3. Let $\mathbf{y}$ = U$\mathbf{x}$. Then solve the equation L$\mathbf{y}$ = $\mathbf{b}$ for $\mathbf{y}$.
4. Take the values for y and solve the equation $\mathbf{y}$ = U$\mathbf{x}$ for $\mathbf{x}$. This will give the solution to the system A$\mathbf{x}$ = $\mathbf{b}$.

Example: Solve the following system using an LU decomposition.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 - 4x_2 + 6x_3 = 18 \\ 3x_1 - 9x_2 - 3x_3 = 6 \end{cases}$$

1. Set up the equation Ax = b.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 - 4x_2 + 6x_3 = 18 \\ 3x_1 - 9x_2 - 3x_3 = 6 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

2. Find an LU decomposition for A. This will yield the equation (LU)$\mathbf{x}$ = $\mathbf{b}$.

*Note: We found the LU composition for A earlier. Its given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So. . .

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}.$$

and $\mathbf{y}$ = U$\mathbf{x}$ → $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

3. Let $\mathbf{y}$ = U$\mathbf{x}$. Then solve the equation L$\mathbf{y}$ = $\mathbf{b}$ for $\mathbf{y}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

where $\mathbf{y}$ = U$\mathbf{x}$ → $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$

Now solving for y gives the following values:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{cases} y_1 = 5 \\ 2y_1 - 8y_2 = 18 \\ 3x_1 - 15y_2 - 12y_3 = 6 \end{cases} \rightarrow \begin{matrix} y_1 = 5 \\ y_2 = -1 \\ y_3 = 2 \end{matrix}$$

4. Take the values for y and solve the equation $\mathbf{y}$ = U$\mathbf{x}$ for $\mathbf{x}$. This will give the solution to the system A$\mathbf{x}$ = $\mathbf{b}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ x_2 = -1 \\ x_3 = 2 \end{cases} \quad \begin{matrix} x_1 = 1 \\ \to x_2 = -1 \\ x_3 = 2 \end{matrix}$$

Therefore, the solution to the system is $x_1 = 1$, $x_2 = -1$, and $x_3 = 2$.