

GRAPH THEORY - TREES

Trees are graphs that do not contain even a single cycle. They represent hierarchical structure in a graphical form. Trees belong to the simplest class of graphs. Despite their simplicity, they have a rich structure.

Trees provide a range of useful applications as simple as a family tree to as complex as trees in data structures of computer science.

Tree

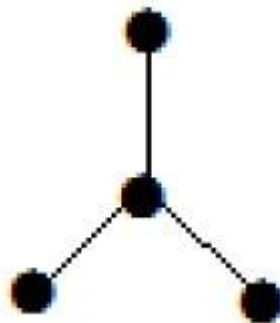
A **connected acyclic graph** is called a tree. In other words, a connected graph with no cycles is called a tree.

The edges of a tree are known as **branches**. Elements of trees are called their **nodes**. The nodes without child nodes are called **leaf nodes**.

A tree with 'n' vertices has 'n-1' edges. If it has one more edge extra than 'n-1', then the extra edge should obviously have to pair up with two vertices which leads to form a cycle. Then, it becomes a cyclic graph which is a violation for the tree graph.

Example 1

The graph shown here is a tree because it has no cycles and it is connected. It has four vertices and three edges, i.e., for 'n' vertices 'n-1' edges as mentioned in the definition.



Note – Every tree has at least two vertices of degree one.

Example 2



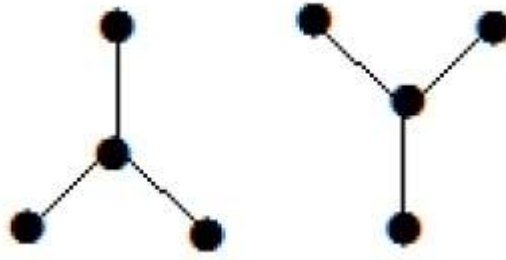
In the above example, the vertices 'a' and 'd' have degree one. And the other two vertices 'b' and 'c' have degree two. This is possible because for not forming a cycle, there should be at least two single edges anywhere in the graph. It is nothing but two edges with a degree of one.

Forest

A **disconnected acyclic graph** is called a forest. In other words, a disjoint collection of trees is called a forest.

Example

The following graph looks like two sub-graphs; but it is a single disconnected graph. There are no cycles in this graph. Hence, clearly it is a forest.



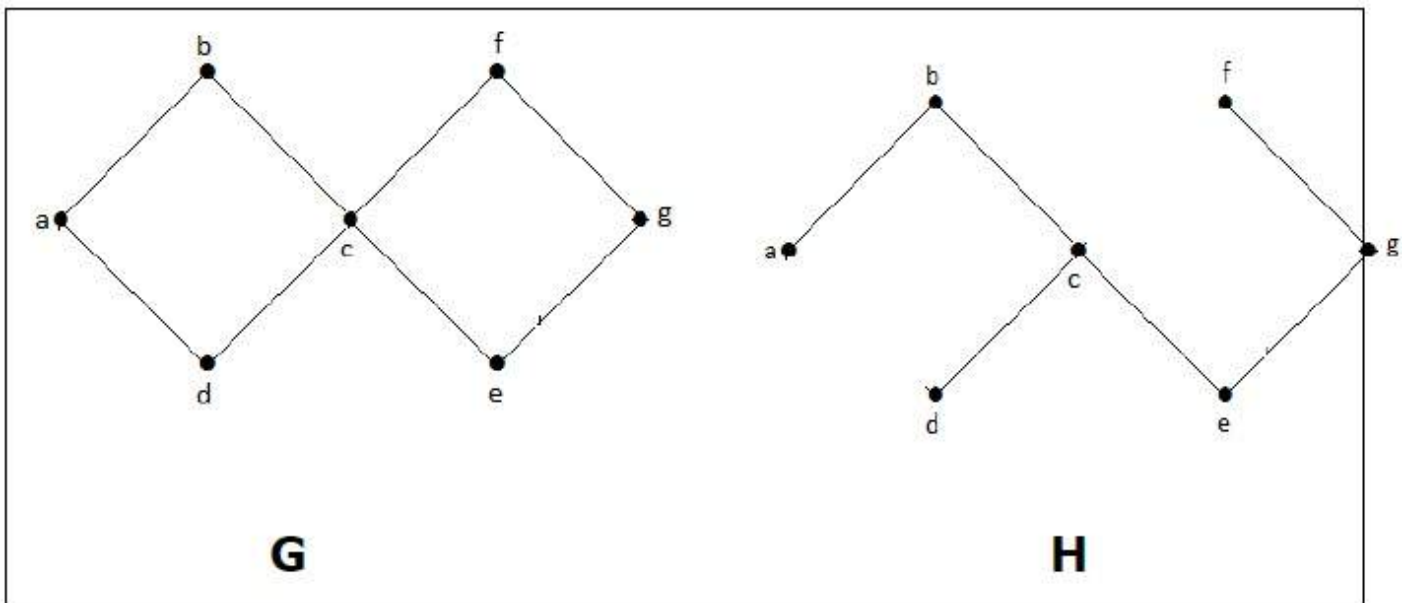
Spanning Trees

Let G be a connected graph, then the sub-graph H of G is called a spanning tree of G if –

- H is a tree
- H contains all vertices of G .

A spanning tree T of an undirected graph G is a subgraph that includes all of the vertices of G .

Example



In the above example, G is a connected graph and H is a sub-graph of G .

Clearly, the graph H has no cycles, it is a tree with six edges which is one less than the total number of vertices. Hence H is the Spanning tree of G .

Circuit Rank

Let ' G ' be a connected graph with ' n ' vertices and ' m ' edges. A spanning tree ' T ' of G contains $n - 1$ edges.

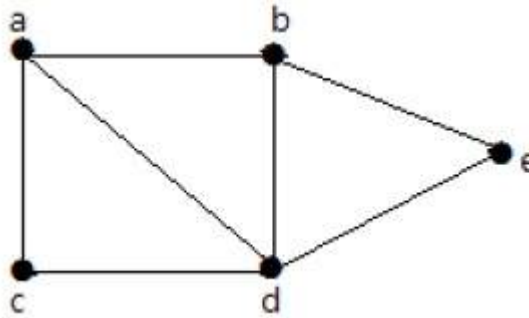
Therefore, the number of edges you need to delete from ' G ' in order to get a spanning tree = $m - n + 1$, which is called the circuit rank of G .

This formula is true, because in a spanning tree you need to have ' $n-1$ ' edges. Out of ' m ' edges, you need to keep ' $n-1$ ' edges in the graph.

Hence, deleting ' $n-1$ ' edges from ' m ' gives the edges to be removed from the graph in order to get a spanning tree, which should not form a cycle.

Example

Take a look at the following graph –



For the graph given in the above example, you have $m=7$ edges and $n=5$ vertices.

Then the circuit rank is

$$\begin{aligned} G &= m - (n - 1) \\ &= 7 - (5 - 1) \\ &= 3 \end{aligned}$$

Example

Let 'G' be a connected graph with six vertices and the degree of each vertex is three. Find the circuit rank of 'G'.

By the sum of degree of vertices theorem,

$$n \sum_{i=1} \deg(V_i) = 2|E|$$

$$6 \times 3 = 2|E|$$

$$|E| = 9$$

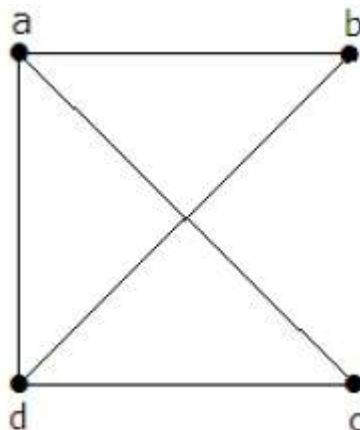
$$\text{Circuit rank} = |E| - |V| + 1$$

$$= 9 - 6 + 1 = 4$$

Kirchoff's Theorem

Kirchoff's theorem is useful in finding the number of spanning trees that can be formed from a connected graph.

Example



The matrix 'A' be filled as, if there is an edge between two vertices, then it should be given as '1', else '0'.

$$\begin{bmatrix} 0 & a & b & c & d \\ a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A = \begin{vmatrix} b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

By using kirchoff's theorem, it should be changed as replacing the principle diagonal values with the degree of vertices and all other elements with -1.A

$$= \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix} = M$$

$$M = \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix}$$

$$\text{Cofactor of } m_{11} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 8$$

Thus, the number of spanning trees = 8.