GRAPH THEORY - CONNECTIVITY

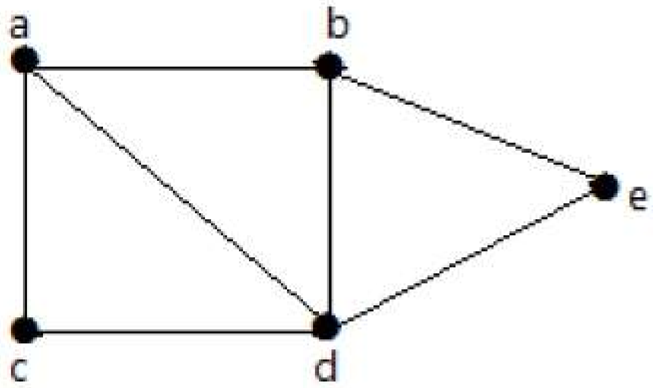
Whether it is possible to traverse a graph from one vertex to another is determined by how a graph is connected. Connectivity is a basic concept in Graph Theory. Connectivity defines whether a graph is connected or disconnected. It has subtopics based on edge and vertex, known as edge connectivity and vertex connectivity. Let us discuss them in detail.

Connectivity

A graph is said to be **connected if there is a path between every pair of vertex**. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

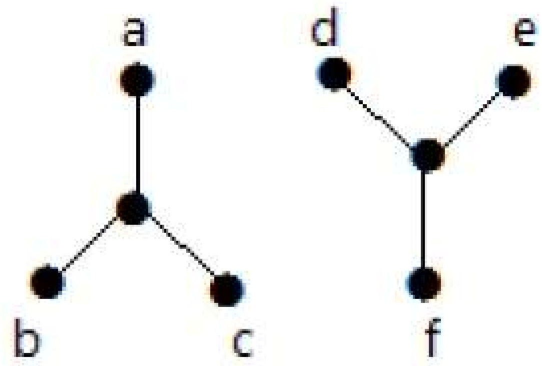
Example 1

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex ‘a’ to vertex ‘e’ using the path ‘a-b-e’.



Example 2

In the following example, traversing from vertex ‘a’ to vertex ‘f’ is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.



Cut Vertex

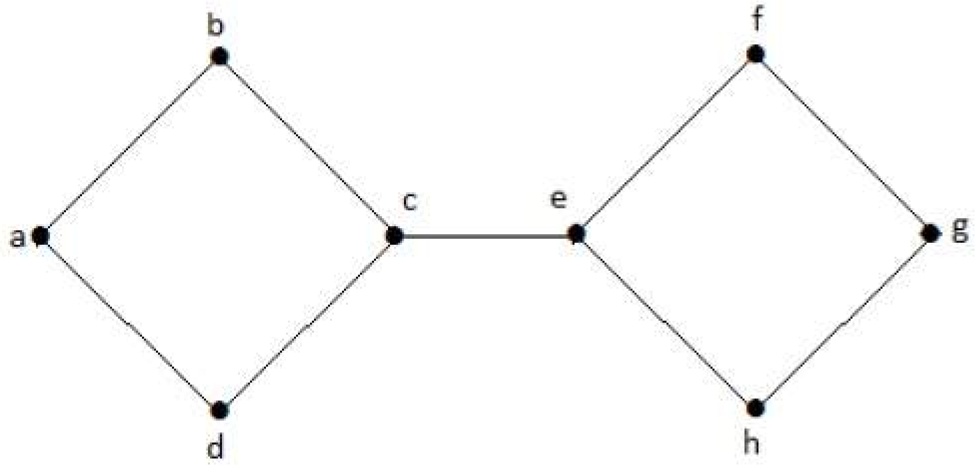
Let ‘G’ be a connected graph. A vertex V ∈ G is called a cut vertex of ‘G’, if ‘G-V’ $Delete‘V'from‘G'$ results in a disconnected graph. Removing a cut vertex from a graph breaks it in to two or more graphs.

Note − Removing a cut vertex may render a graph disconnected.

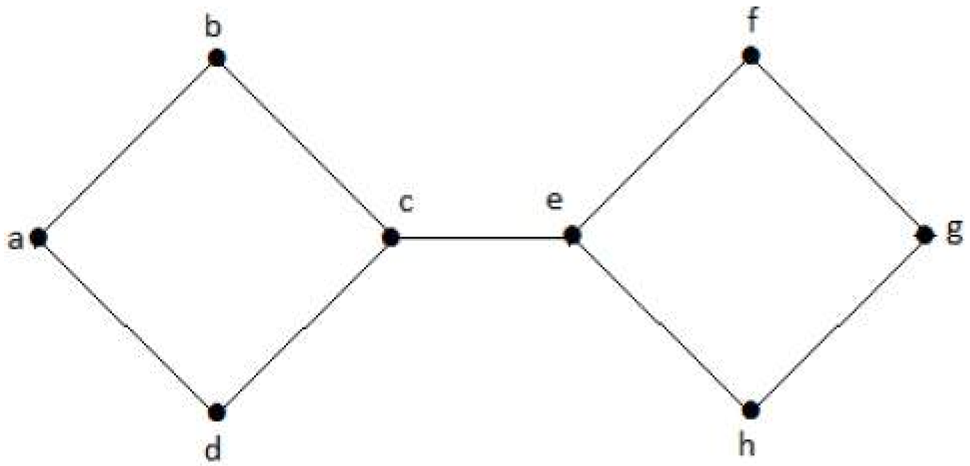
A connected graph ‘G’ may have at most $n-2$ cut vertices.

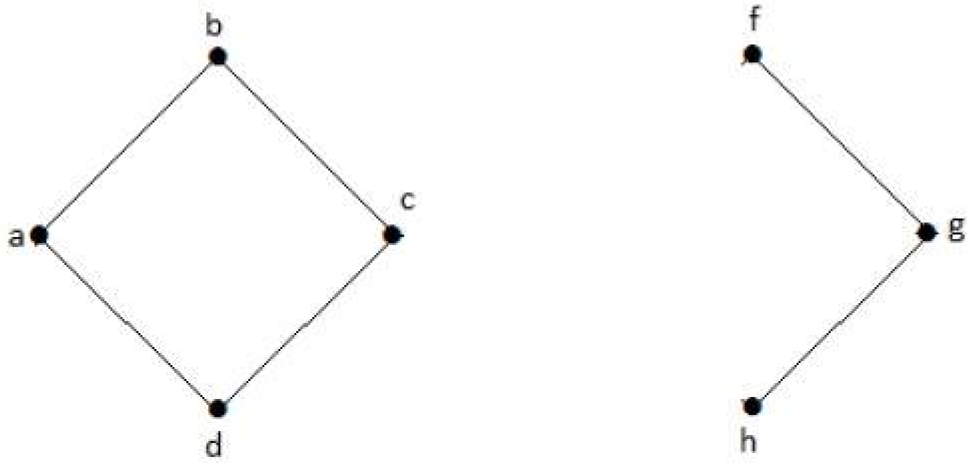
Example

In the following graph, vertices ‘e’ and ‘c’ are the cut vertices.



By removing ‘e’ or ‘c’, the graph will become a disconnected graph.





Without ‘g’, there is no path between vertex ‘c’ and vertex ‘h’ and many other. Hence it is a disconnected graph with cut vertex as ‘e’. Similarly, ‘c’ is also a cut vertex for the above graph.

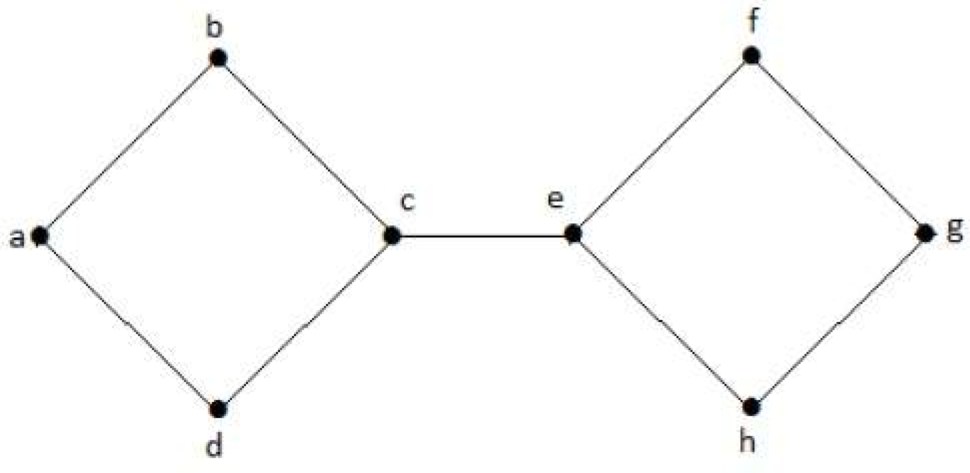
Cut Edge *Bridge*

Let ‘G’ be a connected graph. An edge ‘e’ ∈ G is called a cut edge if ‘G-e’ results in a disconnected graph.

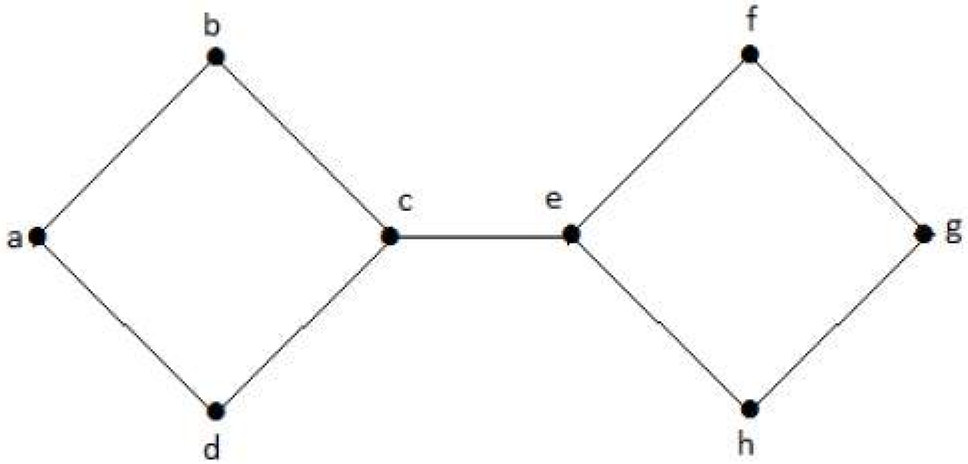
If removing an edge in a graph results in to two or more graphs, then that edge is called a Cut Edge.

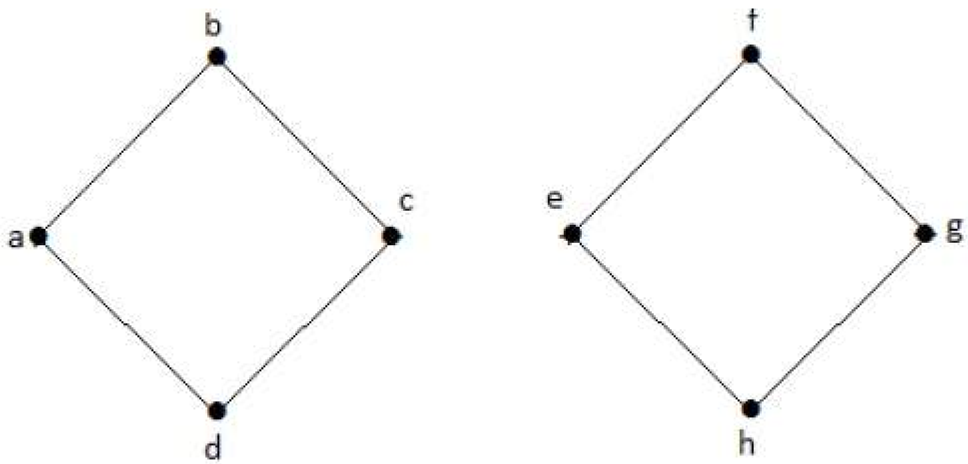
Example

In the following graph, the cut edge is $[c, e]$



By removing the edge c, e from the graph, it becomes a disconnected graph.





In the above graph, removing the edge c, e breaks the graph into two which is nothing but a disconnected graph. Hence, the edge c, e is a cut edge of the graph.

Note − Let 'G' be a connected graph with 'n' vertices, then

- a cut edge e ∈ G if and only if the edge 'e' is not a part of any cycle in G.
- the maximum number of cut edges possible is 'n-1'.
- whenever cut edges exist, cut vertices also exist because at least one vertex of a cut edge is a cut vertex.
- if a cut vertex exists, then a cut edge may or may not exist.

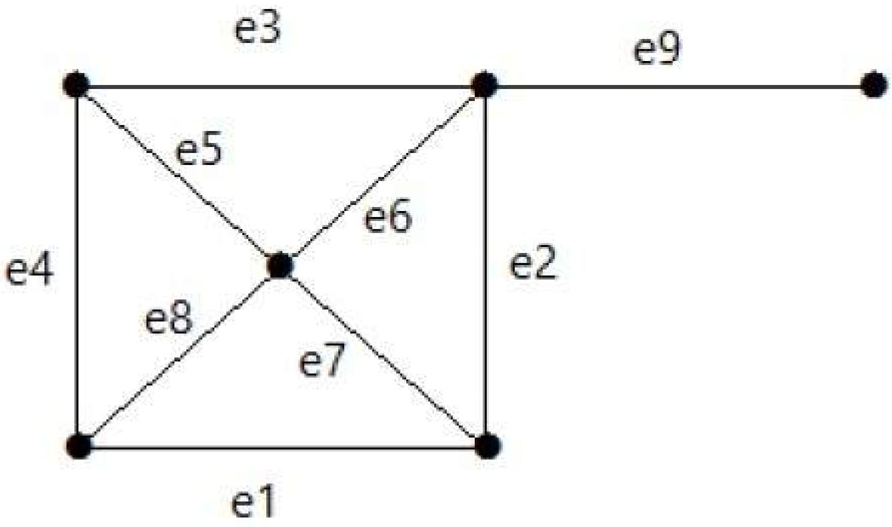
Cut Set of a Graph

Let 'G'= V, E be a connected graph. A subset E' of E is called a cut set of G if deletion of all the edges of E’ from G makes G disconnect.

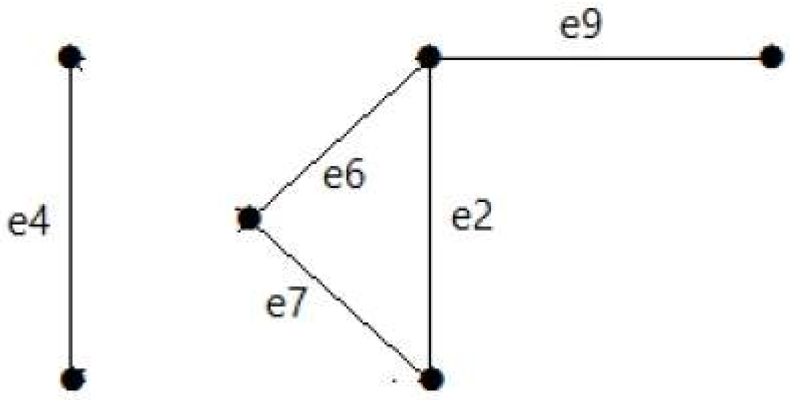
If deleting a certain number of edges from a graph makes it disconnected, then those deleted edges are called the cut set of the graph.

Example

Take a look at the following graph. Its cut set is E1 = {e1, e3, e5, e8}.



After removing the cut set E1 from the graph, it would appear as follows −



Similarly there are other cut sets that can disconnect the graph −

- E3 = {e9} – Smallest cut set of the graph.
- E4 = {e3, e4, e5}

Edge Connectivity

Let ‘G’ be a connected graph. The minimum number of edges whose removal makes ‘G’ disconnected is called edge connectivity of G.

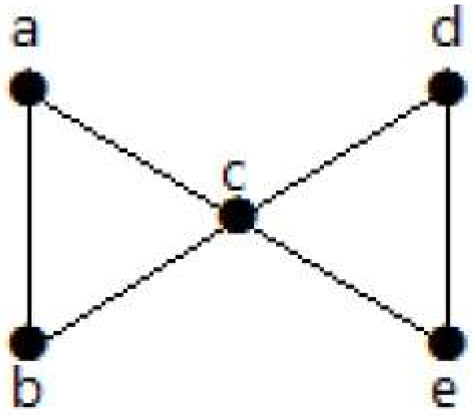
Notation − λG

In other words, the **number of edges in a smallest cut set of G** is called the edge connectivity of G.

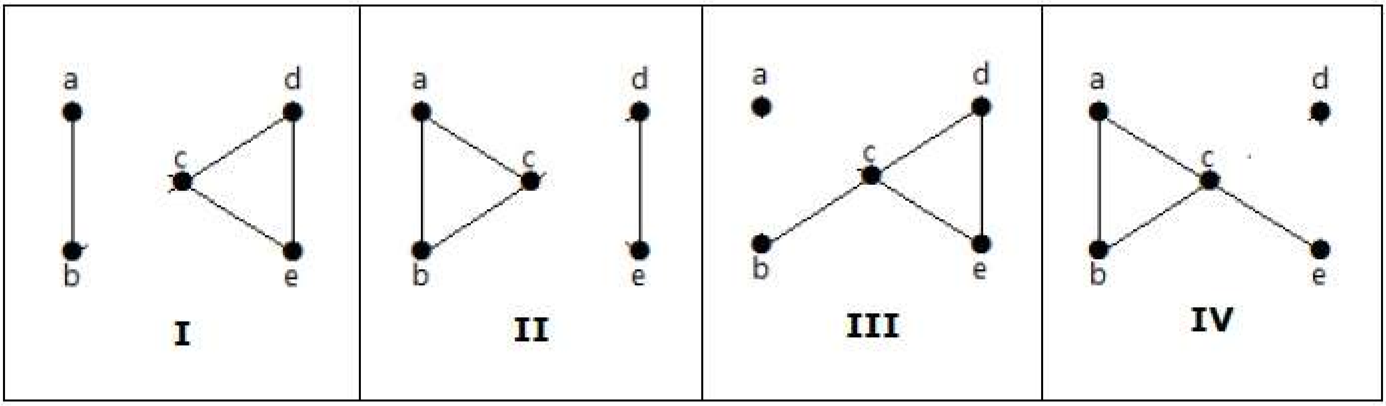
If ‘G’ has a cut edge, then λG is 1. $edgeconnectivityofG.$

Example

Take a look at the following graph. By removing two minimum edges, the connected graph becomes disconnected. Hence, its edge connectivity $\lambda(G)$ is 2.



Here are the four ways to disconnect the graph by removing two edges −



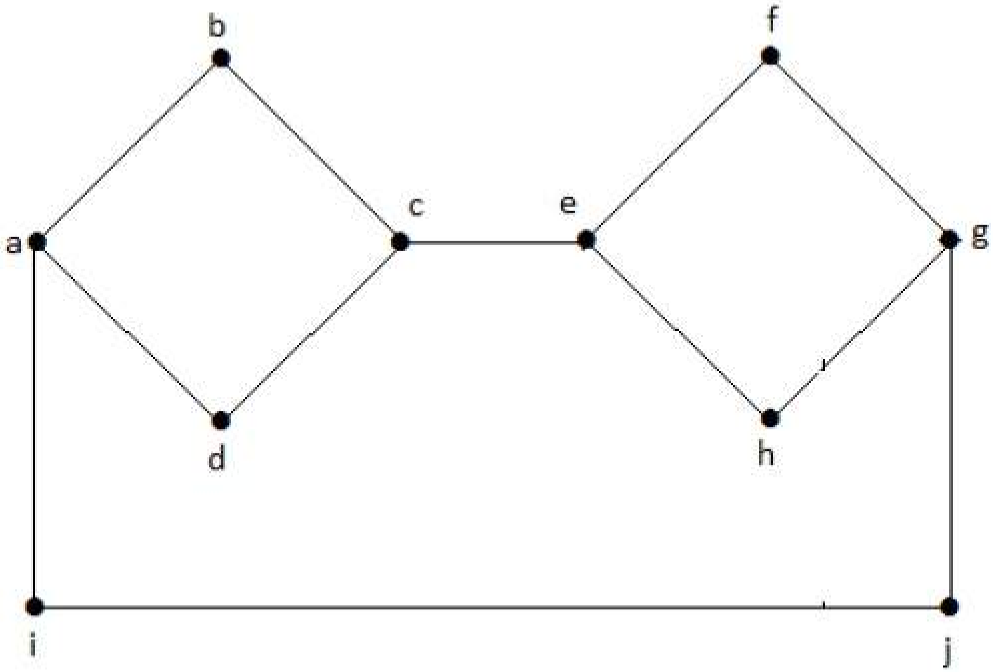
Vertex Connectivity

Let ‘G’ be a connected graph. The minimum number of vertices whose removal makes ‘G’ either disconnected or reduces ‘G’ in to a trivial graph is called its vertex connectivity.

Notation − KG

Example

In the above graph, removing the vertices ‘e’ and ‘i’ makes the graph disconnected.



If G has a cut vertex, then KG = 1.

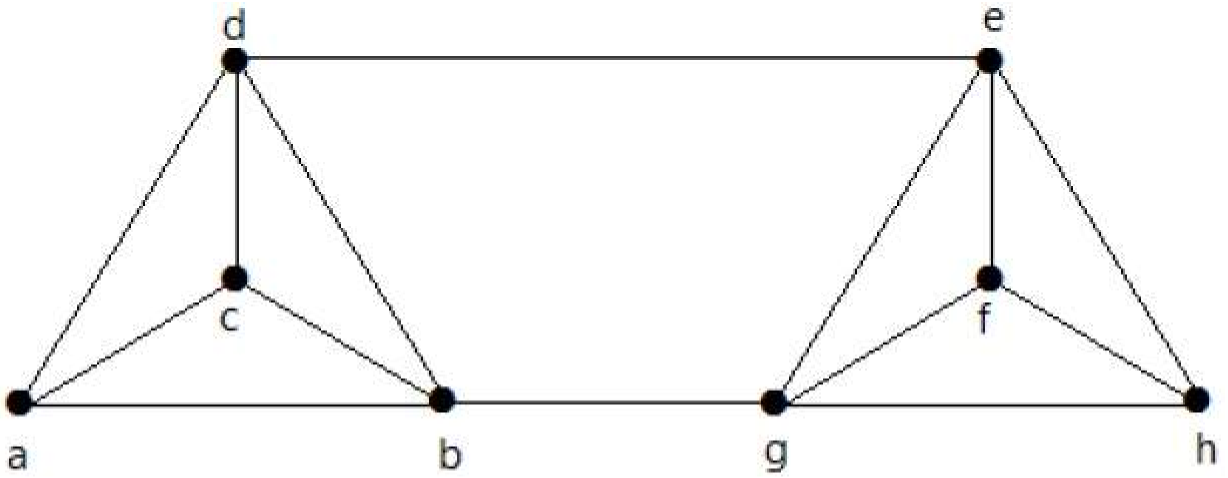
Notation − For any connected graph G,

$$KG \leq \lambda G \leq \delta G$$

Vertex connectivity $K(G)$, edge connectivity $\lambda(G)$, minimum number of degrees of G$\delta(G)$.

Example

Calculate λG and KG for the following graph −



Solution

From the graph,

$\delta G = 3$

$KG \leq \lambda G \leq \delta G = 3$ 1

$KG \geq 2$ 2

Deleting the edges {d, e} and {b, h}, we can disconnect G.

Therefore,

$\lambda G = 2$

$2 \leq \lambda G \leq \delta G = 2$ 3

From 2 and 3, vertex connectivity $KG = 2$