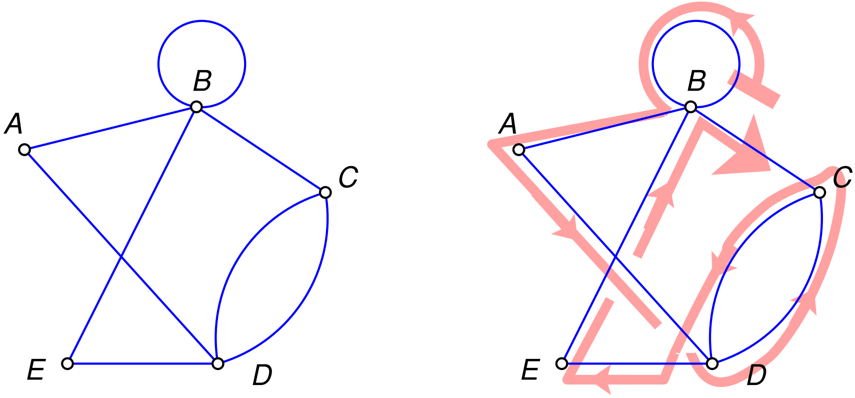
Euler Paths and Euler Circuits

An **Euler path** is a path that uses every edge of a graph exactly once.

An **Euler circuit** is a circuit that uses every edge of a graph exactly once.

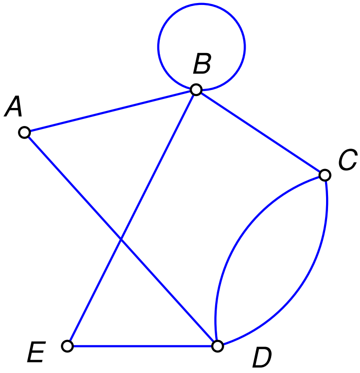
- An Euler path starts and ends at different vertices.
- An Euler circuit starts and ends at the same vertex.

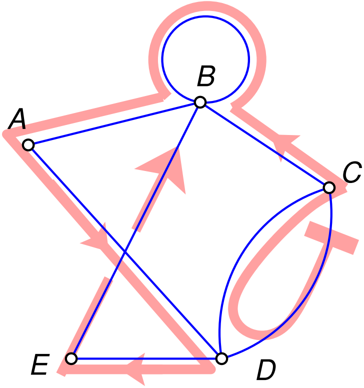
Euler Paths and Euler Circuits



An Euler path: BBADCDEBC

Euler Paths and Euler Circuits





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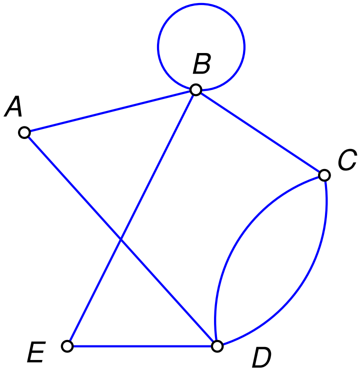

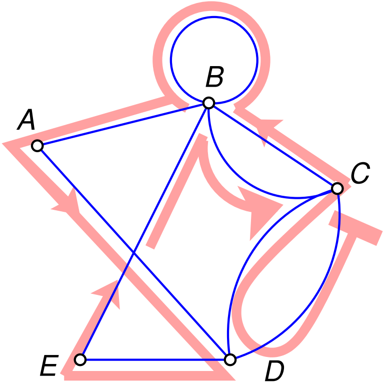

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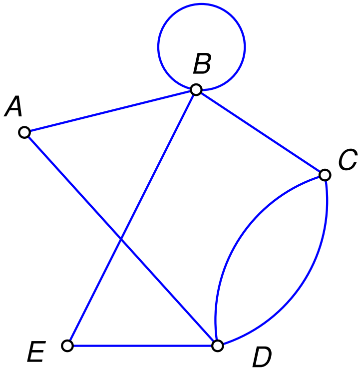

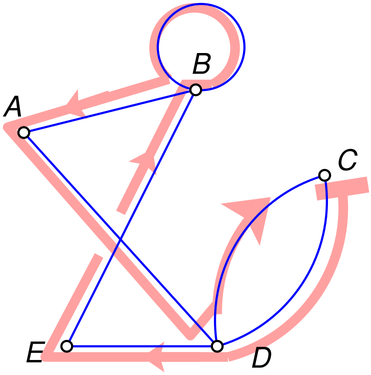

Another Euler circuit: CDEBBADC

Euler Paths and Euler Circuits

Is it possible to determine whether a graph has an Euler path or an Euler circuit, without necessarily having to find one explicitly?

★

The Criterion for Euler Paths

Suppose that a graph has an Euler path P.

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Therefore, all vertices other than the two endpoints of P must be even vertices.

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Then P leaves x one more time than it enters, and leaves y one fewer time than it enters.

Therefore, **the two endpoints of P must be odd vertices.**

The Criterion for Euler Paths

The inescapable conclusion ("based on reason alone!"):

If a graph G has an Euler path, then it must have exactly two odd vertices.

Or, to put it another way,

If the number of odd vertices in G is anything other than 2, then G cannot have an Euler path.

The Criterion for Euler Circuits

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- That is, **v must be an even vertex.**

The Criterion for Euler Circuits

The inescapable conclusion ("based on reason alone"):

If a graph G has an Euler circuit, then all of its vertices must be even vertices.

Or, to put it another way,

If the number of odd vertices in G is anything other than 0, then G cannot have an Euler circuit.

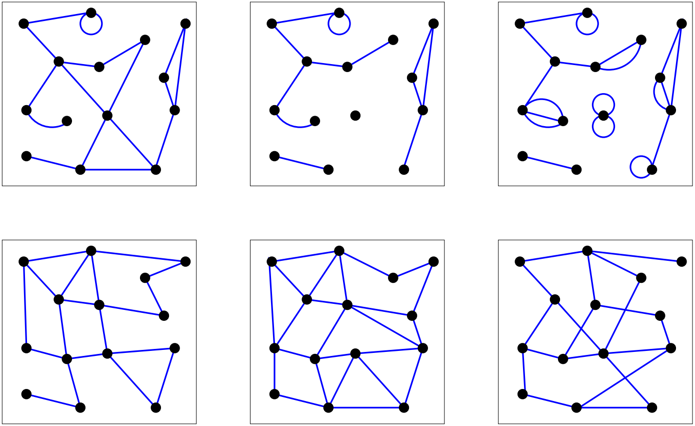
Things You Should Be Wondering

- Does **every** graph with **zero** odd vertices have an Euler circuit?

- Does **every** graph with **two** odd vertices have an Euler path?

- Is it possible for a graph have just **one** odd vertex? ★

How Many Odd Vertices?



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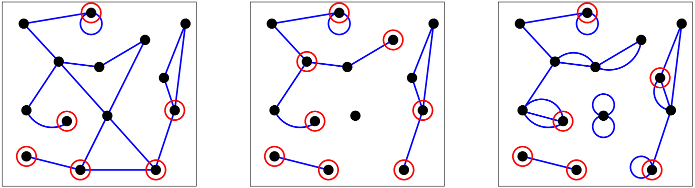

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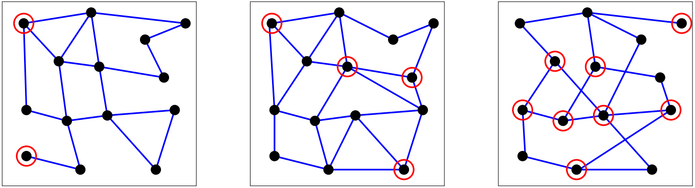

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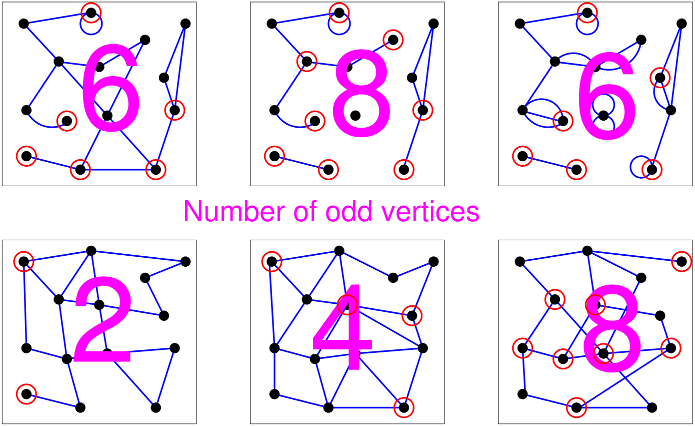

The Handshaking Theorem

The Handshaking Theorem says that

In every graph, the sum of the degrees of all vertices equals twice the number of edges.

If there are n vertices $V_1, \ldots, V_n$, with degrees $d_1, \ldots, d_n$, and there are e edges, then

$$d_1 + d_2 + \cdots + d_{n-1} + d_n = 2e$$

Or, equivalently,

$$e = \frac{d_1 + d_2 + \cdots + d_{n-1} + d_n}{2}$$

The Handshaking Theorem

Why "Handshaking"?

If n people shake hands, and the i^{th} person shakes hands d_i times, then the total number of handshakes that take place is

$$\frac{d_1 + d_2 + \cdots + d_{n-1} + d_n}{2}.$$

(How come? Each handshake involves two people, so the number $d_1 + d_2 + \cdots + d_{n-1} + d_n$ counts every handshake twice.)

The Number of Odd Vertices

- The number of edges in a graph is

$$\frac{d_1 + d_2 + \cdots + d_n}{2}$$

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- **Every graph has an even number of odd vertices!**

Back to Euler Paths and Circuits

Here's what we know so far:

# odd vertices	Euler path?	Euler circuit?
0	No	Maybe
2	Maybe	No
4, 6, 8, ...	No	No
1, 3, 5, ...	*No such graphs exist!*	

Can we give a better answer than "maybe"?

Euler Paths and Circuits — The Last Word

Here is the answer Euler gave:

# odd vertices	Euler path?	Euler circuit?
0	No	**Yes***
2	**Yes***	No
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Euler Paths and Circuits — The Last Word

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Next question: If an Euler path or circuit exists, how do you find it?

Bridges

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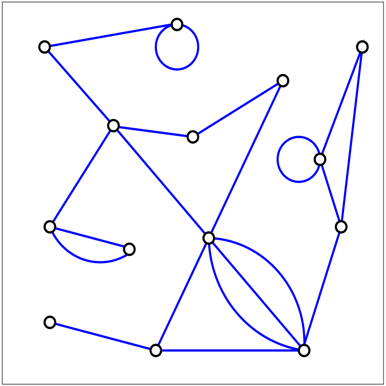

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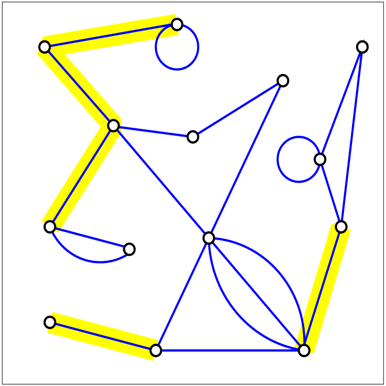
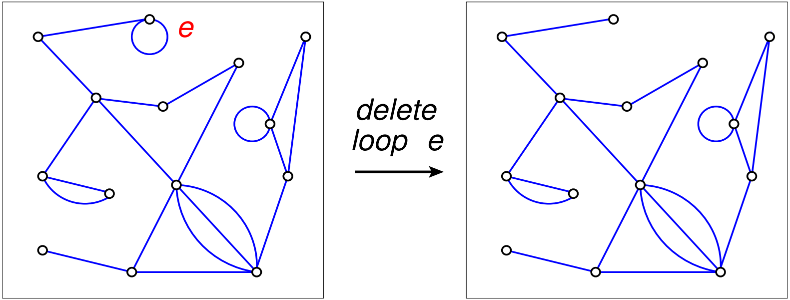


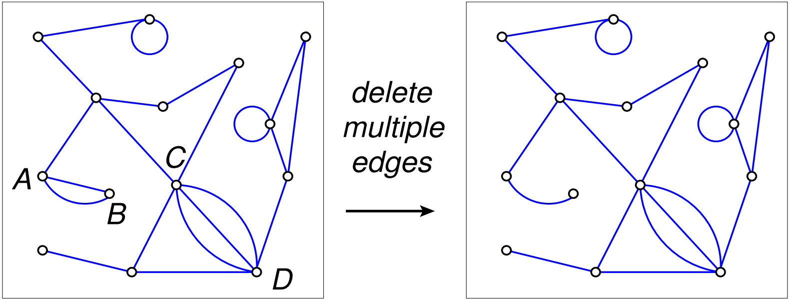
Bridges

Loops cannot be bridges, because removing a loop from a graph cannot make it disconnected.



Bridges

If two or more edges share both endpoints, then removing any one of them cannot make the graph disconnected. Therefore, none of those edges is a bridge.



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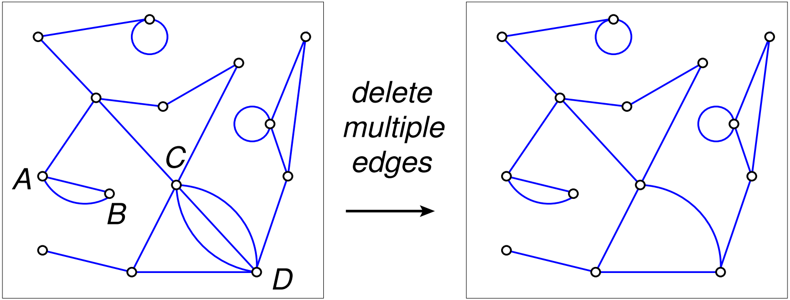
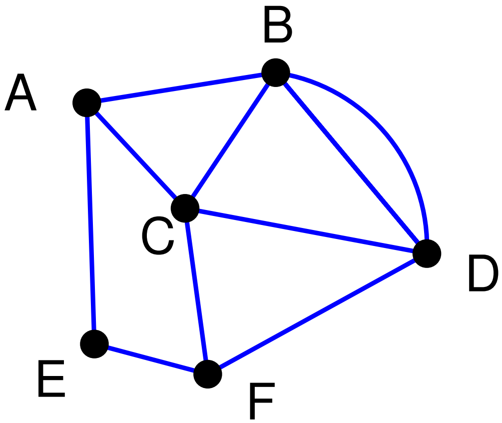


Finding Euler Circuits and Paths

"Don't burn your bridges."

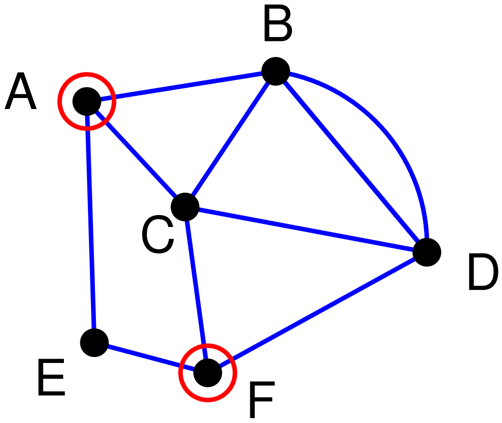
Finding Euler Circuits and Paths

Problem: Find an Euler circuit in the graph below.



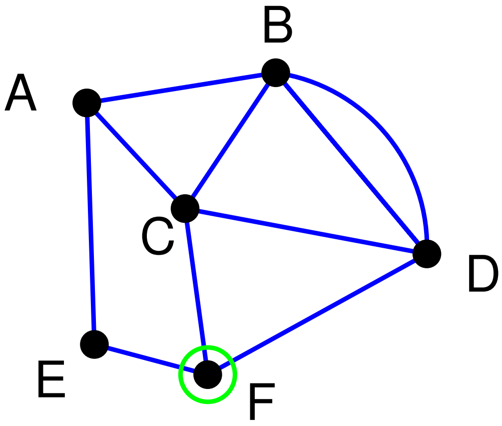
Finding Euler Circuits and Paths

There are two odd vertices, A and F. Let's start at F.



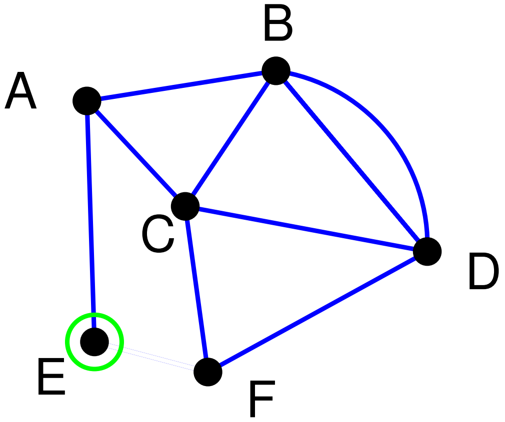
Finding Euler Circuits and Paths

Start walking at F. When you use an edge, delete it.



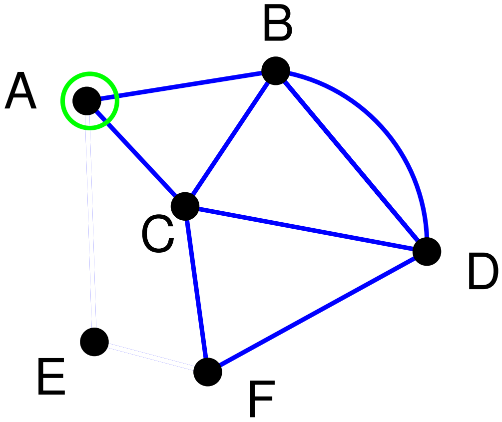
Finding Euler Circuits and Paths

Path so far: FE



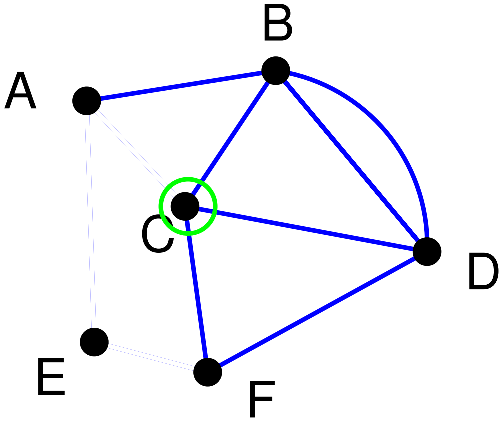
Finding Euler Circuits and Paths

Path so far: FEA



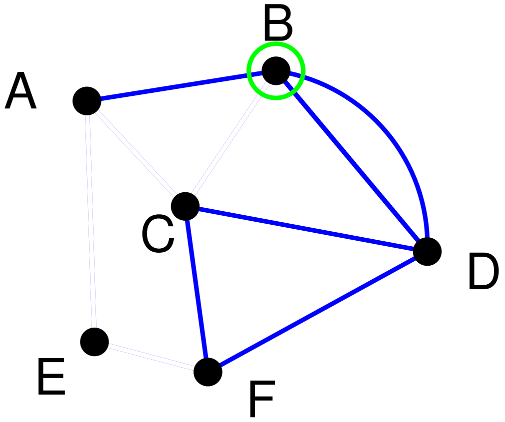
Finding Euler Circuits and Paths

Path so far: FEAC



Finding Euler Circuits and Paths

Path so far: FEACB



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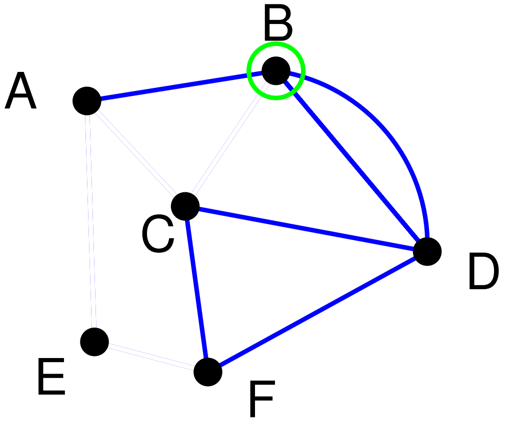
But now, crossing the edge BA would be a mistake, because we would be stuck there.

The reason is that BA is a **bridge**. We don't want to cross ("burn"?) a bridge unless it is the only edge available.

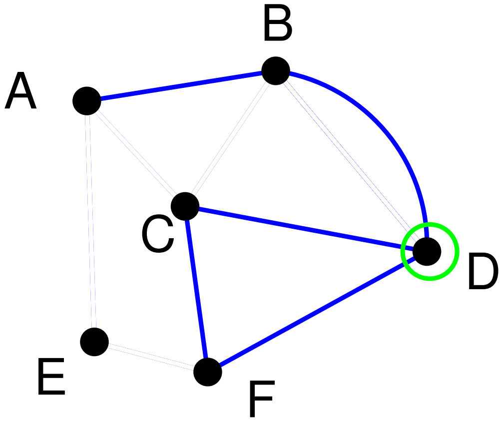
Finding Euler Circuits and Paths

Path so far: FEACB



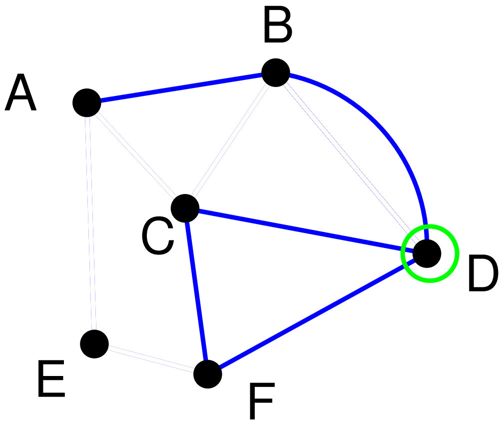
Finding Euler Circuits and Paths

Path so far: FEACBD.



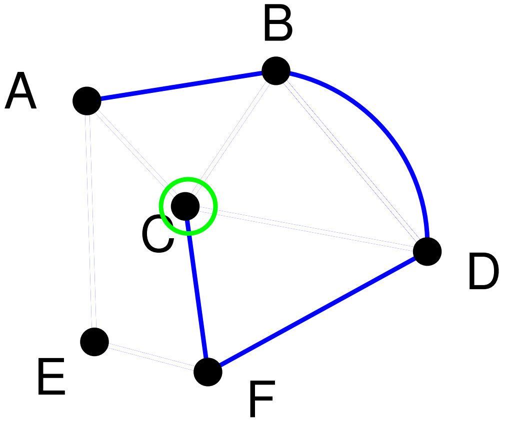
Finding Euler Circuits and Paths

Path so far: FEACBD. Don't cross the bridge!



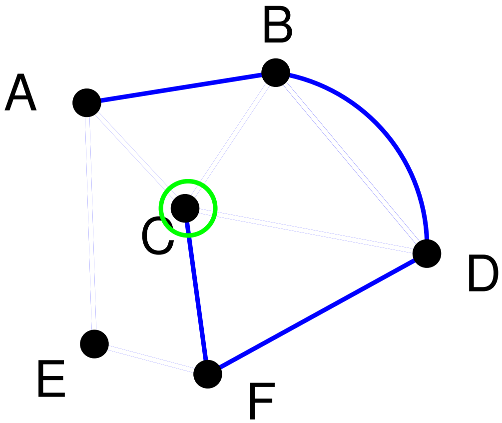
Finding Euler Circuits and Paths

Path so far: FEACBDC



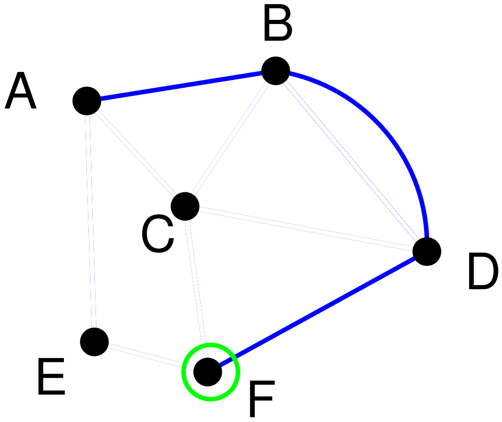
Finding Euler Circuits and Paths

Path so far: FEACBDC   Now we have to cross the bridge CF.



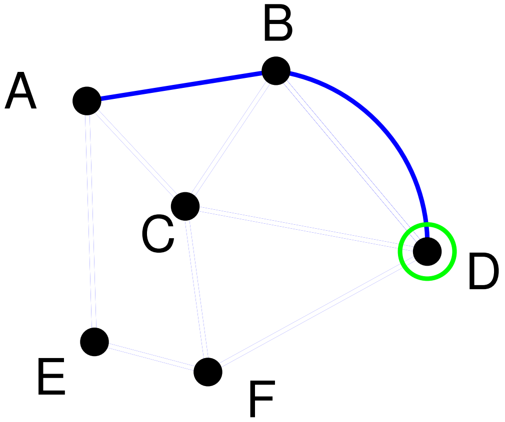
Finding Euler Circuits and Paths

Path so far: FEACBDCF



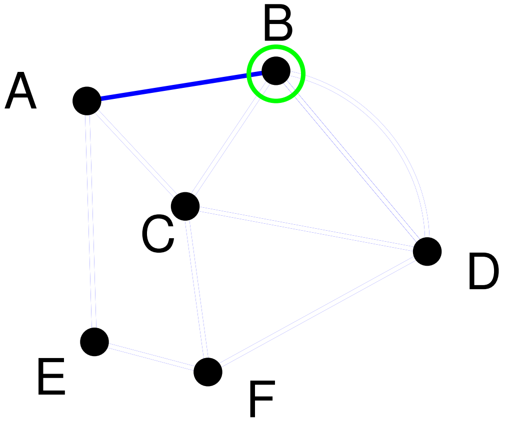
Finding Euler Circuits and Paths

Path so far: FEACBDCFD



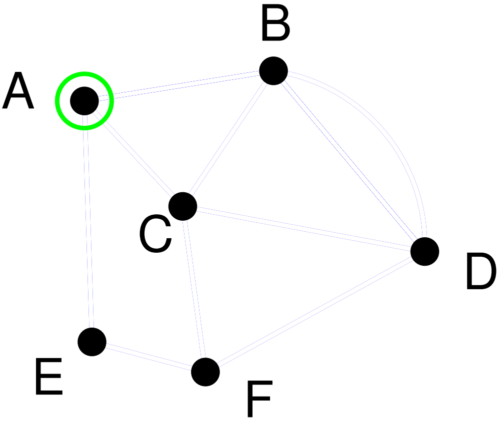
Finding Euler Circuits and Paths

Path so far: FEACBDCFDB



Finding Euler Circuits and Paths

Euler Path: FEACBDCFDBA



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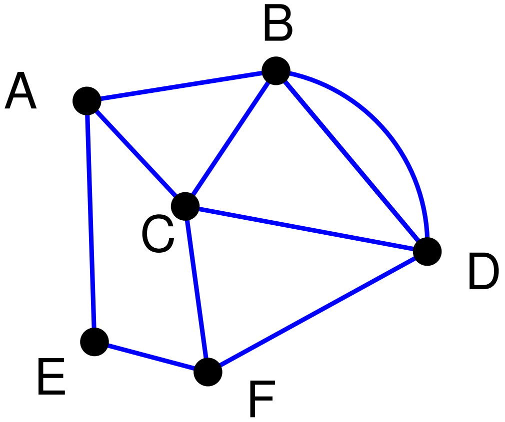

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This is called **Fleury's algorithm**, and it always works!

Fleury's Algorithm: Another Example