Lecture 8.
Basic Concepts of Graph Theory

Undirected and Directed Graphs

Definition

The **undirected graph** (or shortly graph) is a pair:

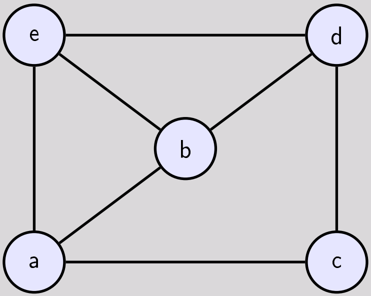
$$G = \langle V, E \rangle$$

where V is a nonempty set (a set of **vertices** of G) and $E = \{\{u, v\} : u, v \in V\}$ is a set of **edges** G.

Example

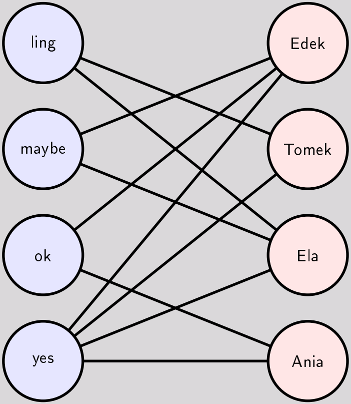
Consider the undirected graph $G = \langle V, E \rangle$, where $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{e, b\}, \{d, b\}, \{c, d\}, \{d, e\}\}$.

We usually use a graphical representation of the graph. The vertices are represented by points and the edges by lines connecting the points.



Example

Consider schools of languages $SH = \{\text{yes}, \text{ok}, \text{maybe}, \text{ling}\}$ and students $S = \{\text{Ania}, \text{Ela}, \text{Tomek}, \text{Edek}\}$ the set of potential students. Every student is interested in some schools $SH_i \subset SH$. The situation can be easily modelled by the graph $G = \langle SH \cup S, E \rangle$.



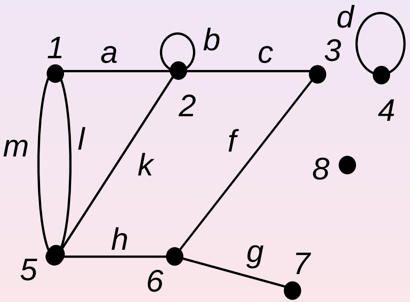
Remark

Let $m = \{p, q\} \in E$. *This means that the edge* m ***connects*** *the vertex* p *and the vertex* q. *Moreover, in such a case* p *and* q *are called the* ***endpoints*** *of* m *and we say, that* p *and* q *are* ***incident*** *with* m *and* p *and* q *are* ***adjacent*** *or* ***neighbours*** *of each other.*

In many applications we use a spacial types of edges: multiedges and loops.

Definition

Multiedges are edges which connect the same pair of vertices and a **loop** connects the vertex with itself. A graph with multiedges and loops are called **multigraph**.



In this picture edges *m* and *l* are multiedges and *d* and *b* are loops.

Often, we associate weights to edges of the graph. These weights can represent cost, profit or loss, length, capacity etc. of given connection.

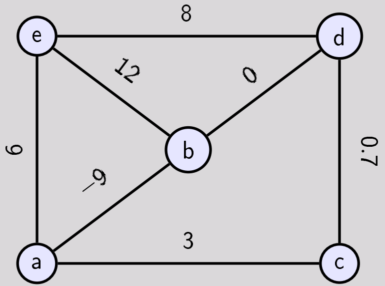
Definition

The **weight** is a mapping from the set of edges to the set of real numbers $w : E \to \mathbb{R}$. The graph with weight function is called the **network** and denoted by $G = \langle V, E, w \rangle$.

Example

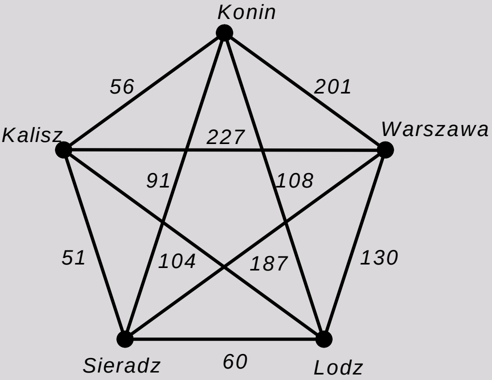
Let us consider the graph $G = \langle V, E, w \rangle$, where $V = \{a, b, c, d, e\}$,
$E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{e, b\}, \{d, b\}, \{c, d\}, \{d, e\}\}$ and the weight function

edge $e \in E$	{a,b}	{a,c}	{a,e}	{d,b}	{e,b}	{d,e}	{d,c}
weight $w(e)$	-9	3	6	0	12	8	0.7



Example

Consider cities Łódź, Warszawa, Konin, Sieradz, Kalisz and road connections between them. Suppose that between any cities there is a direct connection. If the edge is a direct connection, and the weight is the distance, then we obtain the following graph.



Definition

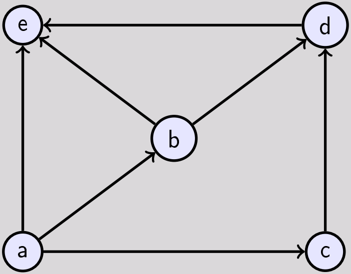
A **directed graph** (**digraph**) is a pair:

$$G = \langle V, E \rangle$$

where V is a nonempty set of vertices and $E = \{(u, v) : u, v \in V\}$ is a set of directed edges. In digraphs, E is collection of ordered pairs. If $(p, q) \in E$, p is the **head** of edge and q is the **tail** of edge.

Example

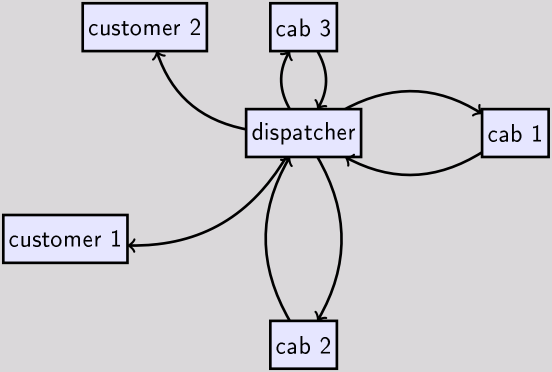
Consider the directed graph $G = \langle V, E \rangle$, where $V = \{a, b, c, d, e\}$,
$E = \{(a, b), (a, c), (a, e), (b, e), (b, d), (c, d), (d, e)\}$.



In the graph theory we can also consider the digraph $\langle V, E, w \rangle$ with the weight function.

Example

A dispatcher for a cab company can communicate in two ways with each cab and one way with a customer. A digraph of this communication model might look as follows.



Definition

The **degree of the vertex** v is the number of edges incident to the vertex, with loops counted twice. The degree of a vertex is denoted $\deg(v)$. A vertex v with degree 0 is called an **isolated vertex**, a vertex with degree 1 is called an **endvertex**, a **hanging vertex**, or a **leaf**.

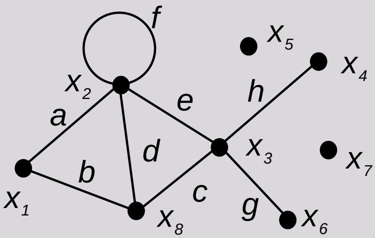
Definition

The **degree of the graph** G, $\Delta(G)$ is the maximum degree of graph G

$$\Delta(G) = \max_{v \in V} \deg(v).$$

Example

Consider the graph



then

1. isolated vertices are x_5 and x_7,
2. the endvertices are x_4 and x_6,
3. $\deg(x_1) = 2$ and $\deg(x_2) = 5$, $\deg(x_3) = 4$ and $\deg(x_8) = 3$
4. degree of graph $\Delta(G) = 5$.

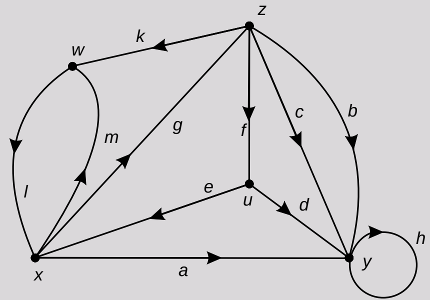
Definition

The **indegree** of a vertex v $degin(v)$, is the number of head endpoints adjacent to v. The **outdegree** of a vertex v $degout(v)$, is the number of tail endpoints adjacent from v. The **degree** of a vertex $deg(v)$ in digraph is equal

$$deg(v) = degout(v) + degin(v)$$

Example

Consider the digraph



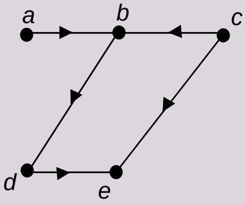
$$\begin{aligned} degout(x) &= 3, & degin(x) &= 2, \\ degout(u) &= 2, & degin(u) &= 1, \\ degout(z) &= 4, & degin(z) &= 1, \\ degout(w) &= 1, & degin(w) &= 2, \\ degout(y) &= 1, & degin(y) &= 5. \end{aligned}$$

Definition

A vertex v for which $degin(v) = 0$ and $degout(v) > 0$ is called a **source** and a vertex v for which $degout(v) = 0$ and $degin(v) > 0$ is called a **sink**.

Example

Consider the digraph

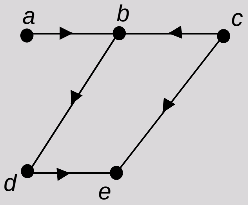


We have the following degrees of vertices

$$deg(a) = degout(a) + degin(a) = 1 + 0 = 1$$
$$deg(b) = degout(b) + degin(b) = 1 + 2 = 3$$
$$deg(c) = degout(c) + degin(c) = 2 + 0 = 2$$
$$deg(d) = degout(d) + degin(d) = 1 + 1 = 2$$
$$deg(e) = degout(e) + degin(e) = 0 + 2 = 2$$

The degree of digraph

$$\Delta(G) = \max_{u \in V} deg(u) = 3$$



Sources in digraph are vertices a and c. The sink is the vertex e.

Theorem

For any graph we have

$$\sum_{v \in V} \deg(v) = 2|E|$$

Theorem

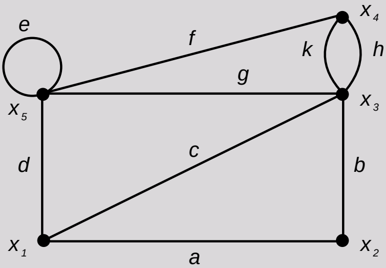
The number of vertices of odd degree is even.

Definition

A **path** in a graph (digraph or multigraph) is a sequence of edges which connect a sequence of vertices. To be more specific, a sequence $(e_1, e_2, \ldots, e_n)$ is called a **path of the length** n if there are vertices $v_0, ..., v_n$ such that $e_i = \{v_{i-1}, v_i\}$ (or $e_i = (v_{i-1}, v_i)$), $i = 1, ..., n$. The first vertex v_1 of the path, is called its **start vertex**, and the last vertex v_n, is called its **end vertex**. Both of them are called terminal vertices of the path. The other vertices in the path are **internal vertices**. If $v_0 = v_n$ then, the sequence is called a **closed path**. A closed path for which the edges and vertices are distinct (except the start and the end vertices) is called a **cycle**.

Example

Consider the graph



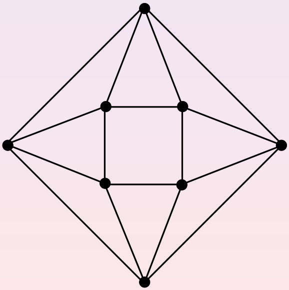
- the path (d, e, g, b, a) is closed path,
- the path (d, g, b, a) is cycle.

Definition

A graph without cycles is called **acyclic**.

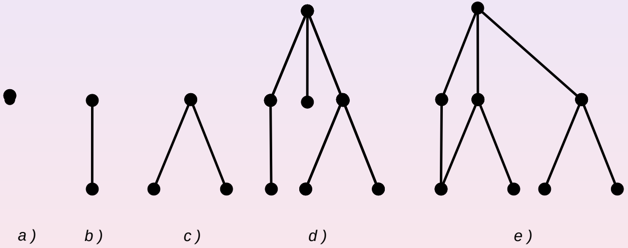
Definition

A graph is called **connected** if for any two vertices u and v there exists a path connecting u and v, i.e. with start vertex u and end vertex v.



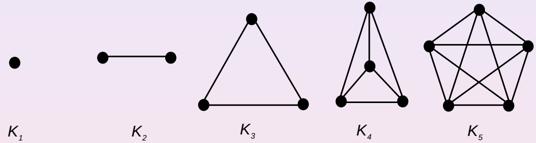
Definition

A connected and acyclic graph is called a **tree**.



Definition

The **complete graph** K_n is the graph with n vertices and all the pairs of vertices are adjacent to each other.



Theorem

In any K_n, $\deg(v) = n - 1$. *The number of edges of any complete graph* K_n *is*

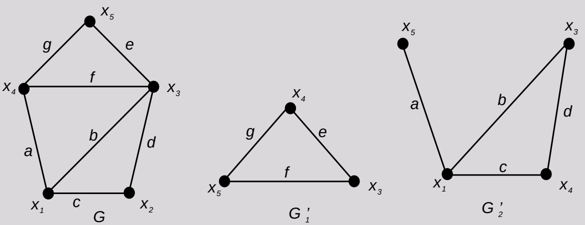
$$|E_{K_n}| = \frac{n(n-1)}{2}.$$

Definition

A graph $H = \langle V_H, E_H \rangle$ is called a **subgraph** of $G = \langle V_G, E_G \rangle$ when $V_H \subset V_G$ and $E_H \subset E_G$.

Example

Consider graphs



G_1 and G_2 are subgraphs of G

$$G_1 \subset G,\ G_2 \subset G$$

Thank you for your attention!