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NONLOCAL FRACTIONAL DIFFERENTIAL INCLUSIONS WITH IMPULSE EFFECTS AND DELAY

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ABSTRACT. Functional fractional differential inclusions with impulse effects in general Banach spaces are studied. We discuss the situation when the semigroup generated by the linear part is equicontinuous and the multifunction is Caratheodory. First, we define the PC-mild solutions for functional fractional semilinear impulsive differential inclusions. We then prove the existence of PC-mild solutions for such inclusions by using the fixed point theorem, multivalued properties and applications of NCHM (noncompactness Hausdorff measure). Eventually, we enhance the acquired results by giving an example.

1. INTRODUCTION

Impulsive differential equations and impulsive differential inclusions have captured tremendous attention in modeling impulsive problems in various areas; physics, technology, optimal control, et cetera. The interest in such equations and inclusions arises essentially from their efficiency in phenomena that cannot be modeled using classical schemes. For instance, processes which change thier state rapidly at certain moments. One can find some applications in [1, 2, 3]. With regard to the basic of general theory on the topic as well as applied developments, see the Benchohra's et al. [4] and the papers [5, 6, 7, 8, 9, 10, 11].

On the other hand, semilinear differential problems with nonlocal conditions are often motivated by empirical problems, for instance see [12] and [13]. The abstarct work of such problems was initiated by Byszewski [13]. Nonlocal problems have been received much concern after it was demonstrated that nonlocal problems can be more descriptive with preferable effects compared to classical ones in applications, see for example [14]. However, handling the operator of solution at zero is considered to be the main obstacle of nonlocal conditions problems in case of studying its compactness. Various techniques and methods have been used by many authors in this direction. For further details, we suggest [6, 7, 8, 9, 10, 11, 15, 16, 17, 18].

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In particularly, Wang et.al. [17] gave a new definition to mild solutions for nonlocal impulsive fractional semilinear differential equations. They investigated the case when the single-valued function is either satisfying Lipschitz condition or a continuous function that takes bounded sets to bounded sets, and addition compactness of the semigroup is assumed in this case. Using NCHM, Li [10] obtained results for nonlocal fractional semilinear differential equations, where the nonlocal term and semigroup are compact and equicontinuous respectivley. Futhermore, Ibrahim and Alsarori [8] determined conditions suffice to guarantee the existence results for the nonlocal impulsive fractional semilinear differential inclusions with delay in case of compactness of the semigroup. Recently, Lian et al. [19] discussed the results of mild solutions existence for nonlocal fractional semilinear inclusions when the operator semigroup is not necessarily compact.

This paper focuses mainly on extending the results of Lian in [19] when the impulse effects and delay are involved. Indeed, under the assumptions that the semigroup and the multivalued function are equicontinuous and Caratheodory respectively we examine the following model:

$$(P_\psi)\begin{cases} {}^cD_t^\alpha x(t) \in Ax(t) + F(t,\tau(t)x),\ t\in J=[0,b],\ t\neq t_i,\\ \Delta x(t_i)=I_i(x(t_i^-)), i=1,...,m,\\ x(t)+g(x)=\psi(t),\ t\in[-r,0],\end{cases}$$

where ${}^cD^\alpha$ denotes the Caputo derivative $(0<\alpha<1)$, A is the infinitesimal generator of the C_0-semigroup $\{T(t), t\geq 0\}$ on E where E is real separable Banach space, $F: J\times\Theta\to 2^E$, $\psi:[-r,0]\to E$, for every $1\leq i\leq m$, $I_i: E\to E$, $g:\Lambda\to E$, and $\Delta x(t_i)=x(t_i^+)-x(t_i^-)$, $x(t_i^+)=\lim_{s\to t_i^+} x(s)$, $x(t_i^-)=\lim_{s\to t_i^-} x(s)$. For any $t\in J$, $\tau(t):\Lambda\to\Theta$, $\tau(t)x(\theta)=x(t+\theta)$, $\theta\in[-r,0]$, $x\in\Lambda$.

2. Preliminaries and Notations

During this section, we state some previous known results so that we can use them later throughout this paper. By $C(J,E)$ we denote the Banach space of all continuous functions on J with $\|x\|=sup\{\|x(t)\|, t\in J\}$, $L^1(J,E)$ is Bochner integrable functions space on J.
Let $P_b(E)$ denote the families of all nonemtpy subsets of E which are bounded, $P_k(E)$ denote the families of all nonemtpy subsets of E which are compact and $P_{ck}(E)$ denote the families of all nonemtpy subsets of E which are convex and compact. Also, $\overline{conv}(B)$ denote the convex closed hull in E of subset B.

Definition 2.1. *([20]). NCHM (noncompactness Hausdorff measure) on E,* $\chi: P_b(E)\to[0,+\infty)$ *is defined by*

$$\chi(B)=inf\{\varepsilon>0: B\subseteq\cup_{j=1}^n B_j\ \ and\ \ radius(B_j)\leq\varepsilon\}.$$

Lemma 2.1. *([20]). Let* χ *as defined above then for any* $B_1, B_2\in P_b(E)$

1. *If* $B_1\subset B_2$ *then* $\chi(B_1)\leq\chi(B_2)$;

2. $\chi(\{a\}\cup B_1)=\chi(B_1)$, *for every* $a\in E$;

3. *For any compact subset* $K\subset E$, $\chi(B_1\cup K)=\chi(B_1)$;

4. $\chi(B_1 + B_2) \leq \chi(B_1) + \chi(B_2)$,
5. $\chi(B_1) = 0$ *iff* B_1 *is relatively compact;*
6. $\chi(tB_1) =| t | \chi(B_1), t \in \mathbb{R}$;
7. $\chi(L(B_1)) \leq \|L\|\chi(B_1)$, *where* L *is a bounded linear operator on* E.

Let $\{t_0, t_1, \cdots, t_m, t_{m+1}\}$ be a partition on $[0, b]$. Let $J_0 = [0, t_1], J_i =]t_i, t_{i+1}]$, $i = 1, \cdots, m$, define

$$PC(J, E) = \{x : J \to E \; and \; x_{|_{J_i}} \in C(J_i, E),\; x(t_i^+) \; and \; x(t_i^-) \; exist,\; 0 \leq i \leq m\}.$$

$$\Theta = \{\psi : [-r, 0] \to E : \psi \textit{ is continuous everywhere except for a finite number of points } t \textit{ at which } \psi(t^-) \textit{ and } \psi(t^+) \textit{exist and } \psi(t) = \psi(t^-)\},$$

and

$$\Lambda = \{x : [-r, b] \to E : x_{|_{[-r,0]}} \in \Theta, x_{|_{J_i}} \in C(J_i, E), and\, x(t_i^+) \; and\, x(t_i^-) \; exist,\; 0 \leq i \leq m\}.$$

Obviously, $PC(J, E)$, Θ and Λ are Banach spaces with norms $\|x\|_{PC(J,E)}$, $\|\psi\|_\Theta$ and $\|x\|_\Lambda$. $\forall x \in \Lambda$ and $t \in J$, $\tau(t)x$, $x \in \Theta$ defined as

$$\tau(t)x(\theta) = x(t + \theta),\; \forall \theta \in [-r, 0].$$

$\tau(t)x$ represents the history of the state time $t - r$ up the present time t.
For any subset $B \subseteq \Lambda$ and for any $i = 0, 1, 2, \cdots, m$, we define

$$B_{|_{\overline{J_i}}} = \{x^* : \overline{J_i} \longrightarrow E : x^*(t) = x(t), t \in J_i, x^*(t_i) = x(t_i^+), x \in B, \}.$$

Also, let us consider the map $\chi_\Lambda : P_b(\Lambda) \to [0, \infty[$ defined by

$$\chi_\Lambda(B) = \chi_\Theta(B_{|_{[-r,0]}}) + \chi_{PC}(B) = \chi_\Theta(B_{|_{[-r,0]}}) + \max_{i=0,1,\cdots,m} \chi_i(B_{|_{\overline{J_i}}}),\; B \in P_b(\Lambda),$$

where χ_i is NCHM on $C(\overline{J_i}, E)$. It is easy to see that χ_Λ is the NCHM on Λ.

Definition 2.2. *We called* $x \in \Lambda$ *is a mild solution of* (P_ψ) *if*

$$x(t) = \begin{cases} \psi(t) - g(x),\; t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \int_0^t (t - s)^{\alpha-1} S_\alpha(t - s) f(s) ds, t \in J_0, \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \sum_{i=1}^{i=m} \mathcal{T}_\alpha(t - t_i) y_i + \int_0^t (t - s)^{\alpha-1} S_\alpha(t - s) f(s) ds, t \in J_i, \end{cases}$$

where $y_i = I_i(x(t_i^-)), i = 1, 2, ..., m$ *,* f *is an integrable selection for* $F(\cdot, \tau(\cdot)x)$*,*
$\mathcal{T}_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha\theta) d\theta$, $S_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha\theta) d\theta$,
$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{\frac{-1}{\alpha}}) \geq 0$, $\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha)$, $\theta \in (0, \infty)$ *and* ξ *is a probability density function defined on* $(0, \infty)$*, that is* $\int_0^\infty \xi_\alpha(\theta) d\theta = 1$.

Lemma 2.2. *(*[6]*) Assume that* $(W_n)_{n \geq 1}$ *is a sequence of nonempty bounded and closed subsets of E which is decreasing, then* $\cap_{n=1}^\infty W_n$ *is compact nonempty subset of* E *provided that* $\chi(W_n) \to 0$ *as* $n \to \infty$.

Lemma 2.3. *(*[21]*).* $\chi(W(t))$ *is continuous function and* $\chi(W) = \sup_{t \in J} \chi(W(t))$*, where* W *be bounded and equicontinuous subset of* $C(J, E)$.

Lemma 2.4. *([22]). Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of uniformly integrable functions in $L^1(J,E)$, $\chi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable, $\chi(\{\int_0^t u_n(s)ds\}_{n=1}^{\infty}) \leq 2\int_0^t \chi(\{u_n(s)\}_{n=1}^{\infty})ds$.*

Lemma 2.5. *([23]). If B is a bounded subset of E, then $\forall\ \epsilon > 0$, $\exists\ \{u_n\}_{n=1}^{\infty}$ in B with $\chi(B) \leq 2\chi(\{u_n\}_{n=1}^{\infty}) + \epsilon$.*

Lemma 2.6. *([24]). Let E be separable Banach space and $W \in P_k(E)$. If $(W_n)_{n\geq 1} \subset W$. Then*

$$\overline{conv}(\limsup_{n\to\infty} W_n) = \cap_{N>0}\ \overline{conv}(\cup_{n\geq N} W_n).$$

Definition 2.3. *([20, 22]). If X , Y are topological spaces. $F : X \to P(Y)$ is called:*
1. Upper semicontinuous $(u.s.c)$ if for any open set $V \subseteq Y$, $F^{-1}(V) \subseteq$ is an open.
2. Completely continuous if $\forall$ bounded set $B \subset X$, $F(B)$ is relatively compact.
3. Closed in case when its graph is closed in the topological space $X \times Y$.
4. F is said to have a fixed point if there is $x \in X$ such that $x \in F(x)$.

Remark 2.1. *For any closed subset U in X, if , $F(x)$ is closed $\forall x \in U$, and $\overline{F(U)}$ is compact. Then F is $u.s.c.$ iff F is closed.*

Definition 2.4. *The multivalued map $F : J \times E \to P(E)$ is said to be Caratheodory if*

(1) *$t \to F(t,x)$ is measurable for each $x \in E$.*
(2) *$x \to F(t,x)$ is upper semicontinuous.*

Definition 2.5. *If $W = \{f_n : n \in \mathbb{N}\} \subset L^1(J,E)$, then W is semi-compact if:*
(i) W is integrably bounded.
(ii) W is relatively compact.

Lemma 2.7. *([20]). If $W = \{f_n : n \in \mathbb{N}\}$ is semi-compact in $L^1(J,E)$, then W is weakly compact in $L^1(J,E)$.*

Theorem 2.1. *([25]). Let $G : \mathcal{M} \to 2^{\mathcal{M}}$ be u. s. c., $\forall x \in \mathcal{M}$, $G(x)$ is a nonempty, convex closed subset of $\mathcal{M}$ where $\mathcal{M}$ is nonempty bounded, convex, closed and compact subset of E. Then $\exists\ x \in \mathcal{M}$ such that $x \in G(x)$.*

3. MAIN RESULTS

Now we will use the NCHM and multivalued fixed point theorem to prove the existence of mild solutions for our problem P_ψ.

Theorem 3.1. *Suppose the following hypotheses:*
(HA) $A : D(A) \subset E \to E$ generates an equicontinuous C_0-semigroup $\{T(t) : t \geq 0\}$. Also, $\exists\ M$ such that $\sup_{t\in J} \|T(t)\| \leq M$, where M is a positive constant.
(HF) $F : J \times \Theta \to P_{ck}(E)$ is a multifunction satisfies:

(1) *F is Caratheodory and for each fixed $x \in \Lambda$ the set $S^1_{F(\cdot,\tau(\cdot)x)} = \{f \in L^1(J,E) : f(t) \in F(t,\tau(t)x),\ a.e.\} \neq \varnothing$.*

(2) *If* $q \in (0, \alpha)$, *there exists* $\varsigma \in L^{\frac{1}{q}}(J, \mathbb{R}^+)$, *with for any* $x \in \Theta$, $\|F(t, x)\| \leq \varsigma(t)(1 + \|x(0)\|)$ *for a.e.* $t \in J$.

(3) *There exists a constant* $L > 0$ *with* $\frac{4MLb^\alpha}{\Gamma(1+\alpha)} < 1$ *such that for any bounded subset* B *of* Θ, *we have* $\chi(F(t, B)) \leq L\chi(B)$ *for a.e.* $t \in J$.

(Hg) Let $g : \Lambda \to E$ *be continuous, compact function and there exists a constant* $\beta > 0$ *such that* $\|g(x)\| \leq \beta$ *for all* $x \in \Lambda$.

(HI) For every $i = 1, 2 \cdots, m$, *let* $I_i : E \to E$ *be continuous and compact function with* $\|I_i(x)\| \leq h_i\|x\| \; \forall \, x \in E$ *where* h_i *is a positive constant.*

(Hr) There is a positive constant r *such that*

$$(M + 1)[\|\psi\| + \beta] + M[h(r + \|\psi\|] + \gamma(1 + \|\psi\| + r)\|\varsigma\|_{L^{\frac{1}{q}}_{([0,t],\mathbb{R}^+)}}] \leq r \tag{3.1}$$

Where $h = \sum_{i=1}^m h_i$, $\gamma = \frac{Mb^{\alpha-q}}{\Gamma(1+\alpha)(\omega+1)^{1-q}}$ *and* $\omega = \frac{\alpha-1}{1-q}$

Then P_ψ *has at least one mild solution.*

Proof. Let $G : \Lambda \to 2^\Lambda$, such that: $y \in G(x)$ iff

$$y(t) = \begin{cases} \psi(t) - g(x), t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, t \in J_0, \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, t \in J_i. \end{cases} \tag{3.2}$$

Where $f \in S^1_{F(\cdot,\tau(\cdot)x)}$ and $1 \leq i \leq m$. Clearly, any fixed point of G is a mild solution for P_ψ. Hence, our aim is to show that G has fixed point by using Theorem 2.1.

Firstly, We show that $G(x) \subset \Lambda$ are convex in Λ. Let $x \in \Lambda$, $y_1, y_2 \in G(x)$ and $\lambda \in (0, 1)$ and let $t \in [-r, 0]$ from (3.2) we have

$$\lambda y_1(t) + (1 - \lambda) y_2(t) = \psi(t) - g(x).$$

Thus, $\lambda y_1(t) + (1 - \lambda) y_2(t) \in G(x), t \in [-r, 0]$. If $t \in J_0$, from (3.2) we have

$$\lambda y_1(t) + (1 - \lambda) y_2(t) = \mathcal{T}_\alpha(t) g(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[\lambda f_1(s) + (1 - \lambda) f_2(s)] ds,$$

where $f_1, f_2 \in S^1_{F(.,\tau(.)x)}$. Easily, one can see that $S^1_{F(.,\tau(.)x)}$ is convex because the values of F are convex. Then, $[\lambda f_1 + (1-\lambda) f_2] \in S^1_{F(.,\tau(.)x)}$. Thus, $\lambda y_1(t) + (1-\lambda) y_2(t) \in G(x), t \in J_0$. Similarly, we can show that $\lambda y_1(t) + (1 - \lambda) y_2(t) \in G(x)$ for $t \in J_i, i = 1, 2, \cdots, m$. Which means that $G(x)$ is convex for each $x \in \Lambda$.

Next we show that G has closed value for every $x \in \Lambda$.
Suppose that $\{z_n\}_{n=1}^{\infty}$ is a sequence in $G(x)$ such that $z_n \to z$ as $n \to \infty$. We need to prove that $z \in G(x)$. From (3.2), there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset S^1_{F(.,\tau(.)x)}$ such that for $i = 1, \cdots, m$

$$z_n(t) = \begin{cases} \psi(t) - g(x),\ t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) ds, t \in J_0, \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) ds, t \in J_i. \end{cases} \tag{3.3}$$

By (HF)(3.2) we have $\forall n \geq 1$ and $a.e.\ t \in J$, $\|f_n(t)\| \leq \varsigma(t)(1 + \|x(t)\|) \leq \varsigma(t)(1 + \|x\|_\Lambda)$. So, $\{f_n : n \geq 1\}$ is integrable bounded. Also, since $\{f_n(t) : n \geq 1\} \subset F(\cdot, \tau(\cdot)x)$, $\{f_n : n \geq 1\}$ is relatively compact in E for $a.e.\ t \in J$. Then, the set $\{f_n : n \geq 1\}$ semicompact. So, it is weakly compact in $L^1(J, E)$. We suppose that the sequence $(f_n)_{n \geq 1}$ converges weakly to $f \in L^1(J, E)$.Thus, there is a sequence (Mazur's Lemma) $\{v_n\}_{n=1}^{\infty} \subseteq \overline{conv}\{f_n : n \geq 1\}$ such that v_n converges strongly to f. As F is u. s. c. with convex and compact values, then by using Lemma 2.7 we get

$$f(t) \in \cap_{k \geq 1} \{\overline{v_n : n \geq k}\}$$
$$\subseteq \cap_{k \geq 1} \overline{conv}\{f_n(t) : n \geq k\} \subseteq F(t, \tau(t)x)$$

Therefore, $f \in S^1_{F(t,\tau(t)x)}$.

Also, by using Holder inequality it can be shown that for all $t \in J, s \in (0, t]$ and $\forall n \geq 1$ we have

$$\|(t-s)^{\alpha-1} S_\alpha(t-s) f_n(s)\| \leq |t-s|^{\alpha-1} \frac{M\alpha}{\Gamma(\alpha+1)} \varsigma(s)(1 + \|x\|_\Lambda) \in L^1(J, \mathbb{R}^+).$$

Therefore, by the Lebesgue dominated convergence theorem, if we take $n \to \infty$ on both sides of (3.3), we will get

$$z(t) = \begin{cases} \psi(t) - g(x), t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, t \in J_0, \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, t \in J_i, 1 \leq i \leq m. \end{cases}$$

Which means that $z(t) \in G(x)$.

Now, let us set $B_0 = \{x \in \Lambda : \|x - x_0\| \leq r, t \in J\}$. Where

$$x_0(t) = \begin{cases} \psi(t), t \in [-r, 0], \\ \psi(0), t \in J. \end{cases}$$

Clearly, B_0 is a bounded subset of Λ. Moreover, B_0 is closed and convex. We need to prove that $G(B_0) \subset B_0$. For fixed $y \in G(B_0)$, let $x \in B_0$ such that $y \in G(x)$ and $t \in J$. If

$t \in [-r, 0]$, then by using (Hg) and (3.1) we have

$$\|y(t) - x_0(t)\| \leq \|g(x)\| \leq \beta \leq r$$

By using Lemma 2.2, (HF)(3.2), (Hg), (3.1) and Holder's inequality we have for $t \in J_0$

$$\|y(t) - x_0(t)\| \leq \|\psi(0)\| + M(\|\psi(0)\| + \beta + \frac{M}{\Gamma(\alpha)}(1 + \|x\|_\Lambda) \int_0^t (t-s)^{\alpha-1}\varsigma(s)ds$$

$$\leq \|\psi\| + M(\|\psi\| + \beta) + \frac{b^{\alpha-q}M(1 + \|\psi\| + r)}{\Gamma(\alpha)(\varpi + 1)^{1-q}}\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}$$

$$\leq \|\psi\| + M(\|\psi\| + \beta) + \gamma(1 + \|\psi\| + r)\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \leq r.$$

Similarly, for $t \in J_i$, $i = 1, \cdots, m$, by using (HI) in addition we get

$$\|y(t) - x_0(t)\| \leq \|\psi\| + M(\|\psi\| + \beta) + h(r + \|\psi\|) + \gamma(1 + \|\psi\| + r)\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \leq r.$$

Which follows that $y \in B_0$. Then, $G(B_0) \subset B_0$.

In this step we show that $G(B_0)_{|\overline{J_i}}$ is equicontinuous, $i = 0, 1, \cdots, m$ where $G(B_0)_{|\overline{J_i}} = \{y^* \in C(\overline{J_i}, E) : y^*(t) = y(t), t \in J_i = (t_i, t_{i+1}], y^*(t_i) = y(t_i^+), y \in G(B_0)\}$.

Let $y \in G(B_0)$. Then there exist $x \in B_0$ with $y \in G(x)$. Form (3.2), there exist $f \in S^1_{F(.,\tau(.)x)}$ such that

$$y(t) = \begin{cases} \psi(t) - g(x),\ t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, t \in J_0, \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, t \in J_i, 1 \leq i \leq m. \end{cases}$$

By the continuity of ψ, one can see easily that if $t, t + \sigma \in [-r, 0]$, then

$$\lim_{\sigma \to 0} \|y^*(t + \sigma) - y^*(t)\| = 0. \tag{3.4}$$

Now, we prove $G(B_0)$ is equicontinuous on J, to show that it is enough to verify the equicontinuity of $G(B_0)_{|\overline{J_i}}$ $\forall i = 0, 1, \cdots, m$. We study two cases:
Case 1. if $i = 0$.

1. Let $t = 0$ and $\sigma > 0$ such that $t + \sigma \in (0, t_1]$. By using Holder's inequality, (HA) and Lemma 2.2(vi), we get

$$\|y^*(t + \sigma) - y^*(t)\| = \|y(\sigma) - y(0)\| \leq \|\mathcal{T}_\alpha(\sigma)(\psi(0) - g(x)) - \mathcal{T}_\alpha(0)(\psi(0) - g(x))\|$$

$$+\| \int_0^{\sigma} (\sigma - s)^{\alpha-1} S_\alpha(\sigma - s) f(s) ds \|$$

$$\leq \|\mathcal{T}_\alpha(\sigma) - \mathcal{T}_\alpha(0)\| \|\psi(0) - g(x)\| + \frac{M(1 + \|\psi\| + r)}{\Gamma(\alpha)} \|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \frac{\sigma^{\alpha-q}}{(\varpi+1)^{1-q}} \to 0 \ as\, \sigma \to 0.$$

Therefore,

$$\lim_{\sigma \to 0} \|y^*(t+\sigma) - y^*(t)\| = 0, \tag{3.5}$$

not dependent on x.

2. Let $t \in (0, t_1)$ and $\sigma > 0$ such that $t + \sigma \in (0, t_1)$, then

$$\|y^*(t+\sigma) - y^*(t)\| = \|y(t+\sigma) - y(t)\| \leq \|\mathcal{T}_\alpha(t+\sigma)(\psi(0) - g(x)) - \mathcal{T}_\alpha(t)(\psi(0) - g(x))\|$$

$$+\| \int_0^{t+\sigma} (t+\sigma-s)^{\alpha-1} S_\alpha(t+\sigma-s) f(s) ds - \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds \|$$

$$\leq G_1 + G_2 + G_3 + G_4,$$

where
$G_1 = \|\mathcal{T}_\alpha(t+\sigma)(\psi(0) - g(x)) - \mathcal{T}_\alpha(t)(\psi(0) - g(x))\|$,
$G_2 = \| \int_0^t [(t+\sigma-s)^{\alpha-1} - (t-s)^{\alpha-1}] S_\alpha(t+\sigma-s) f(s) ds \|$,
$G_3 = \| \int_0^t (t-s)^{\alpha-1} [S_\alpha(t+\sigma-s) - S_\alpha(t-s)] f(s) ds \|$,
$G_4 = \| \int_t^{t+\sigma} (t+\sigma-s)^{\alpha-1} S_\alpha(t+\sigma-s) f(s) ds \|$.
We need show that $G_i \to 0$ as $\sigma \to o\ \forall i = 1, 2, 3, 4$. By arguing as in Step 4 in the proof of Theorem 2 of [26] and Step 3 in the proof of Theorem 4 of [8] we obtain

$$\lim_{\sigma \to 0} \|y^*(t+\sigma) - y^*(t)\| = 0. \tag{3.6}$$

3. When $t = t_1$. Let $\sigma > 0$ and $\delta > 0$ such that $t_1 + \sigma \in J_1$ and $t_1 < \delta < t_1 + \sigma \leq t_2$, then we have

$$\|y^*(t_1+\sigma) - y^*(t_1)\| = \lim_{\delta \to t_1^+} \|y(t_1+\sigma) - y(\delta)\|.$$

From the definition of G, we obtain

$$\|y(t_1+\sigma) - y(\delta)\| \leq \|\mathcal{T}_\alpha(t_1+\sigma)(\psi(0) - g(x)) - \mathcal{T}_\alpha(\delta)(\psi(0) - g(x))\|$$

$$+\sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t_1+\sigma-t_k) I_k(x(t_k^-)) - \mathcal{T}_\alpha(\delta - t_k) I_k(x(t_k^-))\|$$

$$+\| \int_0^{t_1+\sigma} (t_1+\sigma-s)^{\alpha-1} S_\alpha(t_1+\sigma-s) f(s) ds - \int_0^{\delta} (\delta-s)^{\alpha-1} S_\alpha(\delta-s) f(s) ds \|.$$

With similar argument as in the previous way, we have

$$\lim_{\substack{\sigma\to 0\\ \delta\to t_1^+}} \|y(t_1+\sigma)-y(\delta)\| = 0. \tag{3.7}$$

Case 2. If $i = 1, 2, \cdots, m$. With similar argument as in the case 1 we can prove that

$$\lim_{\sigma\to 0} \|y^*(t+\sigma)-y^*(t)\| = 0. \tag{3.8}$$

From (3.5) to (3.8) we get $G(B_0)_{|\overline{J_i}}$ is equicontinuous $\forall i = 0, 1, \cdots, m$, and from (3.4) to (3.8) we conclude that $G(B_0)$ is equicontinuous.
Now, we define a sequence $B_n = \overline{conv}G(B_{n-1})$, $n \geq 1$. From the previous, we have that B_n is nonempty, convex and closed in Λ. Moreover, $B_1 = \overline{conv}G(B_0) \subset B_0$. By induction, $(B_n)_{n\geq 1}$ is decreasing sequence of bounded, closed, convex and equicontinuous subsets of Λ. Put $B = \cap_{n=1}^{\infty} B_n$. So, B is a bounded, closed, convex and equicontinuous subset of Λ and $G(B) \subset B$. We want to prove that B is nonempty and compact in Λ. By light of Lemma 2.2, we need only to show that $\lim_{n\to\infty} \chi_\Lambda(B_n) = 0$, where χ_Λ is the NCHM on Λ. By Lemma 2.5, for arbitrary $\epsilon > 0$ there exist sequence $\{y_k\}_{k=1}^{\infty}$ in $G(B_{n-1})$ such that

$$\chi_\Lambda(B_n) = \chi_\Lambda G(B_{n-1}) \leq 2\chi_\Lambda\{y_k : k \geq 1\} + \epsilon \leq 2\chi_\Theta\{y_k : k \geq 1\} + 2\chi_{PC}\{y_k : k \geq 1\} + \epsilon.$$

From the definition of χ_{PC}, $\chi_\Lambda(B_n) \leq 2\chi_\Theta(v_{|[-r,0]}) + 2\max_{0\leq i\leq m} \chi_i(v_{|\overline{J_i}}) + \varepsilon$, where $v = \{y_k : k \geq 1\}$ and χ_i is the NCHM on $C(\overline{J_i}, E)$. By using the equicontinuity $B_{n|\overline{J_i}}$, $i = 0, 1, \cdots, m$, by using Lemma 2.3 we get

$$\chi_i(v_{|\overline{J_i}}) = \sup_{t\in\overline{J_i}} \chi(v(t)),$$

where χ is NCHM on E. Thus, by using the nonsinglarity of χ we get

$$\chi_\Lambda(B_n) \leq 2\chi_\Theta(v_{|[-r,0]}) + 2\max_{i=0,1,\cdots,m}[\sup_{t\in\overline{J_i}} \chi(v(t))] + \varepsilon = 2\sup_{t\in[-r,0]} \chi(v(t)) + 2\sup_{t\in J} \chi(v(t)) + \epsilon.$$

Then,

$$\chi_\Lambda(B_n) \leq 2\sup_{t\in[-r,0]} \chi(v(t)) + 2\sup_{t\in J} \chi\{y_k : k \geq 1\} + \epsilon. \tag{3.9}$$

Since $y_k \in G(B_{n-1}), k \geq 1$ there is $x_k \in B_{n-1}$ such that $y_k \in G(x_k), k \geq 1$. From the definition of G, there exist $f_k \in S^1_{F(\cdot,\tau(\cdot)x_k(\cdot))}$. So, we can rewrite (3.9) as

$$\chi_\Lambda(B_n) \leq 2\sup_{t\in J} \chi\{y_k : k \geq 1\} \leq \begin{cases} \chi(\psi(t) - g(x_k)), \ \ t \in [-r, 0], \\ \chi(\mathcal{T}_\alpha(t)(\psi(0) - g(x_k))) \\ +\chi(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_k(s)ds), t \in J_0, \\ \chi(\mathcal{T}_\alpha(t)(\psi(0) - g(x_k))) + \sum_{r=1}^{r=i} \chi(\mathcal{T}_\alpha(t-t_r) I_r(x_k(t_r^-))) \\ +\chi(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_k(s)ds), t \in J_i, 1 \leq i \leq m. \end{cases}$$

Since, g and I_i for every $i = 1, 2, \cdots, m$ are compact, by Lemma 2.1 we get $\forall t \in [-r, 0]$ and $k \geq 1$

$$\chi\{\mathcal{T}_\alpha(t)(\psi(t) - g(x_k)) : k \geq 1\} = 0. \tag{3.10}$$

Moreover, $\forall t \in J$

$$\chi\{\mathcal{T}_\alpha(t)(\psi(0) - g(x_k)) : k \geq 1\} = 0, \tag{3.11}$$

$$\chi\{\mathcal{T}_\alpha(t - t_r)I_r(x_k(t_r^-)) : k \geq 1\} = 0. \tag{3.12}$$

Hence, by (3.10), (3.11) and (3.12) for every $t \in J$ we have

$$\chi_\Lambda(B_n) \leq \varepsilon + 2\sup_{t\in J} \chi\{\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_k(s) ds : k \geq 1\}.$$

Now, we will estimate $\chi\{\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_k(s) ds : k \geq 1\}$. By using Lemma 2.1, Lemma 2.4 and (HF)(3.3), we get

$$\begin{aligned}
\chi_\Lambda(B_n) &\leq 4\int_0^t (t-s)^{\alpha-1} \chi\{S_\alpha(t-s) f_k(s) : k \geq 1\} ds + \varepsilon \\
&\leq \frac{4\alpha M}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \chi(F(s, B_{n-1}(s))) ds + \varepsilon \\
&\leq \frac{4\alpha ML}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \chi(B_{n-1}(s)) ds + \varepsilon \\
&\leq \frac{4\alpha ML}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \chi_{PC}(B_{n-1}) ds + \varepsilon \\
&\leq \frac{4\alpha ML}{\Gamma(1+\alpha)} \chi_{PC}(B_{n-1}) \int_0^t (t-s)^{\alpha-1} ds + \varepsilon \\
&\leq \frac{4MLb^\alpha}{\Gamma(1+\alpha)} \chi_{PC}(B_{n-1}) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we find

$$\chi_\Lambda(B_n) \leq \delta \chi_{PC}(B_{n-1}),$$

where $\delta = \dfrac{4ML}{\Gamma(1+\alpha)} < 1$.

Clearly, by means of finite number of steps we can write

$$0 \leq \chi_\Lambda(B_n) \leq \delta^{n-1} \chi_{PC}(B_1).$$

By using (HF)(3.3), if we take the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \chi_\Lambda(B_n) = 0.$$

Thus, $B = \cap_{n=1}^\infty B_n$ is nonempty and compact by Lemma 2.3.

Now, our aim to prove that $G_{|B} : B \to 2^B$ has closed graph. Let $\{x_n\}_{n=1}^{\infty}$ in B, $x_n \to x$ as $n \to \infty$, $y_n \in G(x_n)$ and $y_n \to y$ as $n \to \infty$. We need to prove that $y \in G(x)$. Because, $y_n \in G(x_n)$, for any $n \geq 1$ there exists $f_n \in S^1_{F(\cdot,\tau(\cdot)x_n)}$ such that

$$y_n(t) = \begin{cases} \psi(t) - g(x_n), t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x_n)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) ds, t \in J_0, \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x_n)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x_n(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) ds, t \in J_i. \end{cases}$$

Since, $x_n \to x$ uniformly, we have for any $t \in J$

$$\lim_{n\to\infty} \|\tau(t)x_n - \tau(t)x\|_\Theta = 0.$$

We know that $\forall n \geq 1, \|f_n(t)\| \leq \varsigma(t)(1 + \|\tau x_n(0)\|) \leq \varsigma(t)(1 + \|x_n\|_\Lambda) \leq \varsigma(t)(1 + r + \|\psi\|)$ for a.e. $t \in J$. This show that $\{f_n : n \geq 1\}$ is integrably bounded. Moreover, (HF)(3.3) and convergence of $\{x_n\}_{n=1}^{\infty}$ implies that

$$\chi\{f_n : n \geq 1\} \leq \chi(F(t, \{x_n(t) : n \geq 1\}) \leq L\chi\{x_n(t) : n \geq 1\} = 0.$$

This means that the sequence $\{f_n : n \geq 1\}$ is relatively compact in E for a.e. $t \in J$. So, the sequence $\{f_n : n \geq 1\}$ is semicompact and by using Lemma 2.7 it is weakly compact in $L^1(J, E)$. We can suppose that $f_n \to f \in L^1(J, E)$ weakly. Then from Mazur's Lemma, there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq \overline{conv}\{f_n : n \geq 1\}$ such that u_n converges strongly to f. Since F is $u.s.c.$ and has compact and convex values, so by using Lemma 2.6 we get

$$f(t) \in \cap_{k\geq 1} \overline{\{u_n(t) : n \leq k\}} \subseteq \cap_{k\geq 1} \overline{conv}\{f_n : n \geq k\}$$

$$\subseteq \cap_{k\geq 1} \overline{conv}\{\cup_{n\geq k} F(t, \tau(t)x_n)\} = \overline{conv} \lim_{n\to\infty} \sup F(t, \tau(t)x_n) \subseteq F(t, \tau(t)x).$$

Then, by continuity of $\psi, g, \mathcal{T}_\alpha, S_\alpha, I_i (i = 1, \cdots, m)$ and by the same arguments used when proving G is closed, we get

$$y(t) = \begin{cases} \psi(t) - g(x),\ t \in [-r, 0], \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, t \in J_0, \\ \mathcal{T}_\alpha(t)(\psi(0) - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, t \in J_i, 1 \leq i \leq m. \end{cases}$$

Hence, $y \in G(x)$. This means that $G_{|B}$ has closed graph.

Finally, we prove that G is $u.s.c.$ on B.
From the above, we have that B is closed and $G(x)$ is closed $\forall x \in B$. Moreover, G is closed and the set $\overline{G(B)} \subseteq B$ is compact. Therefore, by Remark 2.1, we conclude that G is $u.s.c.$.
At the end, by Theorem 2.1, G has at least one point x such that $x \in G(x)$ which is mild solution for the problem P_ψ.

4. EXAMPLE

We study the following system :

$$\begin{cases} \partial_t^\alpha y(t,z) \in \partial_z^2 y(t,z) + R(t,\tau(t,z)y), t \in [0,1], t \neq t_i, z \in [0,1] \\ y(t,0) = y(t,1) = 0, \\ y((\dfrac{i}{m+1})^+, z) = y(\dfrac{i}{m+1}, z) + \dfrac{1}{2^i}, i = 1, \cdots, m, z \in [01], \\ y(t,z) = \sum_{j=0}^{j=q} \int_0^1 k_j(z,v) tan^{-1}(y(s_j,v))dv, z \in [0,1], \end{cases} \tag{4.1}$$

where $0 < s_0 < s_1 < \cdots < s_q < 1$, $k_j \in C([0,1] \times [0,1], \mathbb{R})$, $j = 0, 1, \cdots, q, \partial_t^\alpha$ is the Caputo fractional partial derivative of order α, where $0 < \alpha < 1$ and $R : [0,1] \times E \to P(E)$.

To rewrite (4.1) in the abstract form, we set $E = L^2([-1,1], \mathbb{R})$, and A is the Laplace operator, i.e. $A = \dfrac{\partial^2}{\partial z^2}$ on the domain $D(A) = \{x \in E : x, x'$ are absolutely continuous, and $x'' \in E, x(0) = x(1) = 0\}$. From [27], A is the infinitesimal generator of an analytic and compact semigroup $\{T(t)\}_{t \geq 0}$ in E. This leads to that A satisfies the assumption (HA).
For every $i = 1, \cdots, m$ define $I_i : E \to E$ by

$$I_i(x)(z) = \frac{1}{2^i}, z \in [0,1].$$

Note that the assumption (HI) is satisfied.
For every $j = 0, 1, \cdots, q$, define $H_j : E \to E$ as

$$(H_j(x))(z) = \int_0^1 k_j(z,v) tan^{-1}(x(v))dv, z \in [0,1].$$

Now take $g : PC([0,1], E) \to E$ as

$$g(x) = \sum_{j=0}^{j=q} H_j(x(s_j)).$$

Finally, let $F(t,x)(z) = R(t, x(z))$ and $x(t)(z) = x(t,z)$, where $z \in [0,1]$. Then, the system (4.1) takes the form :

$$\begin{cases} {}^cD^\alpha x(t) \in Ax(t) + F(t, \tau(t)x), t \in J = [0,1], t \neq t_i, i = 1, ..., m, \\ x(t_i^+) = x(t_i) + I_i(x(t_i^-)), i = 1, ..., m, \\ x(t) = g(x), \ \ t \in [-1, 0]. \end{cases}$$

If we put some conditions on F as in Theorem 3.1, then (4.1) has at least one mild solution on $[-1,1]$.

CONCLUSION

The present article discussed the necessary conditions to ensure the existence of mild solutions for nonlocal differential inclusions with impulse effects and delay in Banach space. We investigated the case when the linear part generates a semigroup not required to be compact and the multifunction is Caratheodory. We first defined the PC-mild solutions for functional fractional semilinear impulsive differential inclusions. Then, methods and results of NCHM, multivalued functions and fixed point theorems were used to prove the results. Our results given in this paper developed and extended some previous studies. In the end, an example was presented to support the main findings.

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