Properties of some -Hilfer fractional Fredholm-type integro-differential equations

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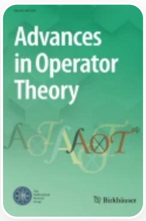

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Abstract

In this paper, we study some properties of Fredholm-type fractional integro-differential equations. For obtaining the results, ψ-Hilfer fractional derivative and ψ Riemann–Liouville integral operator are used. The existence and uniqueness of solution are studied using fixed point theorem and ψ-fractional Bielecki-type norm, and the properties such as estimates and continuous dependence of solution are studied using ψ-fractional Gronwall type of inequalities.

1 Introduction

In this paper, we are concerned with the Fredholm-type ψ-Hilfer fractional integro-differential equation of the type

$$\mathfrak{u}\left(t\right)=\mathfrak{w}\left(t\right)+\frac{1}{\Gamma\left(\alpha\right)}\int\limits_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(t\right)-\psi\left(\tau\right)\right)^{\alpha-1}\mathfrak{g}\left(t,\tau,\mathfrak{u}\left(\tau\right),{}^{H}D_{a+}^{\alpha,\beta;\psi}\mathfrak{u}\left(\tau\right)\right)\mathrm{d}\tau,$$

(1)

for $0<a<b<\infty$, where $\mathfrak{g}\in C(\Delta^{2}\times\mathbb{R}^{n}\times\mathbb{R}^{n},\mathbb{R}_{+})$ and $\mathfrak{g}\in C(\Delta,\mathbb{R}_{+})$; here, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_{+}=[0,\infty)$ denotes the given subset of $\mathbb{R}$, $\mathbb{R}^{n}$ denotes the n-dimensional Euclidean space and $\Delta=[a,b](0<a<b<\infty)$ denotes a finite interval on $\mathbb{R}^{+}$ and C$[a,b]$ denotes the space of continuous function on Δ .

During last few decades, theory of fractional calculus has attracted lot of attention due to its applications in various fields [2, 8, 9]. Since then, various definitions of fractional calculus have appeared in the literature [6]. Recently [1, 13], the authors have proved some fractional Gronwall-like inequality using ψ and ψ-Hilfer fractional definitions and its applications in proving some properties of Cauchy's type of problems. The ψ-fractional calculus is more generalization of various fractional derivative and integral operators. In [4, 3, 7, 14, 17], authors have studied various type of differential and integral equations and its properties using ψ-Hilfer fractional operators. Various results on stability properties can be found in [5, 11, 12]. Recently in [15, 16], authors have obtained the interesting results by investigating the Leibniz rule for ψ-Hilfer fractional derivative.

Motivated by the above work in this paper, we have investigated the existence and uniqueness and other properties such as estimate and continuous dependence of Fredholm type of integro-differential equations using ψ-Hilfer fractional and ψ fractional integral definitions. The good deal of information of various types of fractional differential and integral equations can be found in [6, 9].

2 Preliminaries

Now, we give some basic definitions and lemmas which are needed in proving our further results.

The fractional integrals of a function with respect to another function are defined as follows [6, 9]:

Definition 2.1

[1, 6]. Let $\Delta = [a, b]$ be an interval, $\alpha > 0$, f an integrable function defined on Δ and $\psi \in \mathcal{C}^1(\Delta)$ an increasing function such that $\psi'(x) \neq 0$ for all $x \in \Delta$, then fractional derivative and integral of f are given by

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) \left(\psi(x) - \psi(t)\right)^{\alpha-1} f(t)\, \mathrm{d}t.$$

Similarly, right fractional integral and right fractional derivative are given by

$$I_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) \left(\psi(t) - \psi(x)\right)^{\alpha-1} f(t)\,\mathrm{d}t.$$

In [10], authors have defined ψ-Hilfer fractional derivative as follows:

Definition 2.2

Let $n-1<\alpha<n$ with $n \in N$, $\Delta = [a,b]$ be the interval such that $-\infty \le a < b \le \infty$ and $y, \psi \in C^n(\Delta, \mathbb{R})$ two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in \Delta$. The ψ-Hilfer derivative (left sided and right sided) ${}^H D_{a+}^{\alpha,\beta;\psi}(.)$ of a function of order α and type $0 \le \beta \le 1$ is defined by

$${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{\mathrm{d}}{\mathrm{d}x}\right)^n I_{a+}^{(1-\beta)(1-\alpha):\psi} f(x)$$

and

$${}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{\mathrm{d}}{\mathrm{d}x}\right)^n I_{b-}^{(1-\beta)(1-\alpha):\psi} f(x).$$

In particular when $0<\alpha<1$ and $0 \le \beta \le 1$, we get

$${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma(\gamma-\alpha)} \int_a^x \left(\psi(x) - \psi(t)\right)^{\gamma-\alpha-1} \mathbb{D}_{a+}^{\gamma;\psi} f(t)\,\mathrm{d}t,$$

with $\gamma = \alpha + \beta(1-\alpha)$, and $\mathbb{D}_{a+}^{\gamma;\psi}(.)$ is ψ-Riemann–Liouville fractional derivative.

Now as in [14], we define the (α, ψ, ξ)-Bielecki type of norm:

For $\mathfrak{w}, {}^H D_{a+}^{\alpha,\beta;\psi} \mathfrak{w}(t) \in C(\Delta)$, denote

$$|\mathfrak{w}(t)|_1 = |\mathfrak{w}(t)| + \left|{}^H D_{a+}^{\alpha,\beta;\psi} \mathfrak{w}(t)\right|.$$

(2)

Let B be the space of continuous functions $\mathfrak{w}, {}^H D_{a+}^{\alpha,\beta;\psi} w(t) \in C(\Delta)$ which fulfill the following condition: There exists $K > 0$ such that

$$|\mathfrak{w}(t)|_1 \leq K E_\alpha \left(\xi(\psi(t) - \psi(a))^\alpha\right),$$

(3)

for $t \in \Delta$ where ξ is positive constant and E_α is one-parameter Mittag–Leffler function as given in [14] . In space B, we define the norm

$$|\mathfrak{w}|_B = \sup_{t\in\Delta} \left\{ \frac{|\mathfrak{w}(t)|_1}{E_\alpha \left(\xi(\psi(t) - \psi(a))^\alpha\right)} \right\}.$$

(4)

Then, clearly $(B, \|.\|_E)$ is a Banach space. From Eq. (3), we have

$$|\mathfrak{w}|_B \leq K.$$

(5)

In [13], the authors have proved the following Gronwall's inequality for ψ-Hilfer operator as follows:

Lemma 2.1

[13, Thm 3, p 92]. *Let u, v be two integrable functions and g continuous, with domain* $[a, b]$. *Let* $\psi \in C[a, b]$ *be an increasing function such that* $\psi'(t) \neq 0$ *for* $t \in [a, b]$. *Assume that*

1. *u and v are nonnegative,*

2. *g is nonnegative and non-decreasing.*

$$u(t) \leq v(t) + g(t)\int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau)\,\mathrm{d}\tau,$$

(6)

then

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha k-1} v(\tau)\,\mathrm{d}\tau,$$

(7)

for $t \in [a, b]$.

Now, we give the lemma proved in [10]

Lemma 2.2

[10, Thm 5, p 78]. *If* $f \in C^n[a, b]$, $n-1 < \alpha < n$ *and* $0 \leq \beta \leq 1$, *then*

$$I_{a+}^{\alpha;\psi}\, {}^{H}D_{a+}^{\alpha,\beta;\psi} f(x) = f(x) - \sum_{k=1}^{n} \frac{(\psi(x) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f_{\psi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(a)\,.$$

Lemma 2.3

[10, Thm 7, p 80]. *Let* $f \in C[a, b]$, $\alpha > 0$ *and* $0 \leq \beta \leq 1$, *we have* ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f(x) = f(x)$ and ${}^{H}D_{b-}^{\alpha,\beta;\psi} I_{b-}^{\alpha;\psi} f(x) = f(x)$.

3 Existence and uniqueness

Now, our following theorem is concerned with the existence of unique solution of Eq. (1).

Theorem 3.1

Assume that the following holds:

(i) *the function* $\mathfrak{g}$ *in Eq.* (1) *and its Hilfer fractional derivative with respect to t satisfy the condition*

$$\left|\mathfrak{g}\left(t,\tau,q_1,q_2\right)-\mathfrak{g}\left(t,\tau,\overline{q_1},\overline{q_2}\right)\right|\le c_1\left(t,\tau\right)\left[\left|q_1-\overline{q_1}\right|+\left|q_2-\overline{q_2}\right|\right],$$

(8)

$$\left|{}^{H}D_{a+}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,q_1,q_2\right)-{}^{H}D_{a+}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,\overline{q_1},\overline{q_2}\right)\right|$$
$$\le c_2\left(t,\tau\right)\left[\left|q_1-\overline{q_1}\right|+\left|q_2-\overline{q_2}\right|\right],$$

(9)

and $a\le s\le t\le b, c_1\left(t,\tau\right), c_2\left(t,\tau\right)\in C\left(\Delta^2,\mathbb{R}_+\right)$.

(ii) *there exists nonnegative constants* λ_1,λ_2 such that $\lambda_1+\lambda_2<1$ *and*

$$\frac{1}{\Gamma\left(\alpha\right)}\int_a^b\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}c_1\left(t,\tau\right)E_\alpha\left(\xi\left(\psi\left(\tau\right)-\psi\left(a\right)\right)\right)\mathrm{d}\tau$$
$$\le\lambda_1E_\alpha\left(\xi\left(\psi\left(\tau\right)-\psi\left(a\right)\right)\right),$$

(10)

$$\frac{1}{\Gamma\left(\alpha\right)}\int_a^b\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}c_2\left(t,\tau\right)E_\alpha\left(\xi\left(\psi\left(\tau\right)-\psi\left(a\right)\right)\right)\mathrm{d}\tau$$
$$\le\lambda_2E_\alpha\left(\xi\left(\psi\left(\tau\right)-\psi\left(a\right)\right)\right).$$

(11)

for $t\in\Delta$ *where* ψ *is as given in* (3).

(iii) *there exists nonnegative constants* λ_3,λ_4 *such that*

$$|\mathfrak{w}(t)| + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \mathfrak{g}(t, \tau, 0, 0)\, \mathrm{d}\tau \leq \lambda_3 E_\alpha\left(\xi\left((\psi(t) - \psi(a))^\alpha\right)\right), \tag{12}$$

$$\left|{}^H D_a^{\alpha,\beta;\psi} \mathfrak{w}(t)\right| + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \left|{}^H D_a^{\alpha,\beta;\psi} \mathfrak{g}(t, \tau, 0, 0)\right| \mathrm{d}\tau \leq \lambda_4 E_\alpha\left(\xi\left((\psi(t) - \psi(a))^\alpha\right)\right), \tag{13}$$

where $\mathfrak{w}$, $\mathfrak{g}$ are as defined in (1) and ψ as in (3). Then, Eq. (1) has a unique solution $\mathfrak{u}(t)$ in B.

Proof

Let $\mathfrak{u}(t) \in B$ and define the operator T as

$$(T\mathfrak{u})(t) = \mathfrak{w}(t) + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \mathfrak{g}\left(t, \tau, \mathfrak{u}(\tau), {}^H D_a^{\alpha,\beta;\psi} \mathfrak{u}(\tau)\right) \mathrm{d}\tau. \tag{14}$$

Now taking ψ-Hilfer fractional derivative of (14) with respect to t, we have

$$\left({}^H D_a^{\alpha,\beta;\psi} T\mathfrak{u}\right)(t) = {}^H D_a^{\alpha,\beta;\psi} \mathfrak{w}(t) + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}\, {}^H D_a^{\alpha,\beta;\psi} \mathfrak{g}\left(t, \tau, \mathfrak{u}(\tau), {}^H D_a^{\alpha,\beta;\psi} \mathfrak{u}(\tau)\right) \mathrm{d}\tau.$$

(15)

We show that $T\mathfrak{u}$ maps B into itself.

Since $(T\mathfrak{u})$, $({}^H D_a^{\alpha,\beta;\psi})(T\mathfrak{u})$ are continuous on Δ from (14) and hypotheses, we have

$$\begin{aligned} |(T\mathfrak{u})(t)| \le |\mathfrak{w}(t)| + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1} \left| g\left(t,\tau,\mathfrak{u}(\tau), {}^H D_a^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right)\right. \\ \left. -\mathfrak{g}(t,\tau,0,0) + \mathfrak{g}(t,\tau,0,0)\right| \mathrm{d}\tau \le |\mathfrak{w}(t)| \\ + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1} |\mathfrak{g}(t,\tau,0,0)| \,\mathrm{d}\tau \\ + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1} c_1(t,\tau) |\mathfrak{u}(t)|_1 \mathrm{d}\tau \\ \le \lambda_3 E_\alpha\left(\xi\left((\psi(t)-\psi(a))^\alpha\right)\right) \\ + |\mathfrak{u}|_s \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1} c_1(t,\tau) E_\alpha\left(\xi\left((\psi(t)-\psi(a))^\alpha\right)\right) \\ \le [\lambda_3 + \lambda_1 K] E_\alpha\left(\xi\left((\psi(t)-\psi(a))^\alpha\right)\right), \end{aligned}$$

(16)

and

$$\begin{aligned}\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\left(T\mathfrak{u}\right)(t)\right| &\leq \left|{}^{H}D_{a}^{\alpha,\beta;\psi}w(t)\right| \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,u(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right)\right. \\ &\quad\left.-{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}(t,\tau,0,0)+{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}(t,\tau,0,0)\right|\mathrm{d}\tau \\ &\leq \left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(t)\right|+\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}(t,\tau,0,0)\right|\mathrm{d}\tau \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}c_{2}(t,\tau)\left|u(t)\right|_{1}\mathrm{d}\tau \\ &\leq \lambda_{4}E_{\alpha}\left(\xi\left((\psi(t)-\psi(a))^{\alpha}\right)\right) \\ &\quad+\left|\mathfrak{u}\right|_{E}\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}c_{2}(t,\tau)E_{\alpha}\left(\xi\left((\psi(t)-\psi(a))^{\alpha}\right)\right) \\ &\leq \left[\lambda_{4}+\lambda_{2}K\right]E_{\alpha}\left(\xi\left((\psi(t)-\psi(a))^{\alpha}\right)\right).\end{aligned}$$

(17)

From (16) and (17), we have

$$\begin{aligned}\left|(T\mathfrak{u})(t)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}(T\mathfrak{u})(t)\right| &= \left|(T\mathfrak{u})(t)\right|_{1} \\ &\leq \left[\lambda_{3}+\lambda_{4}+K(\lambda_{1}+\lambda_{2})\right]E_{\alpha}\left(\xi\left((\psi(t)-\psi(a))^{\alpha}\right)\right).\end{aligned}$$

(18)

From (18), it follows that $T\mathfrak{u}\in B$. This proves that T maps B into itself.

Now, we prove that the operator T is a contraction map. Let $\mathfrak{u},v\in B$ from (14) and (15) and hypotheses, we have

$$\begin{aligned}\left|(T\mathfrak{u})(t)-(Tv)(t)\right| &\le \frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|\mathfrak{g}\left(t,\tau,\mathfrak{u}(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right)\right.\\ &\quad\left.-\mathfrak{g}\left(t,\tau,v(\tau),{}^H D_a^{\alpha,\beta;\psi}v(\tau)\right)\right|\mathrm{d}\tau\\ &\le \frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}c_1(t,\tau)\left|\mathfrak{u}(\tau)-v(\tau)\right|_1\mathrm{d}\tau\\ &\le \left|\mathfrak{u}-v\right|_s\frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}c_1(t,\tau)E_\alpha\left(\xi(\psi(t)-\psi(a))^\alpha\right)\\ &\le \left|\mathfrak{u}-v\right|_s\lambda_1 E_\alpha\left(\xi(\psi(t)-\psi(a))^\alpha\right),\end{aligned}$$

(19)

and

$$\begin{aligned}&\left|{}^H D_a^{\alpha,\beta;\psi}(T\mathfrak{u})(t)-{}^H D_a^{\alpha,\beta;\psi}(Tv)(t)\right|\\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|D_a^{\alpha,\beta;\psi}g\left(t,\tau,\mathfrak{u}(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right)\right.\\ &\qquad\left.-D_a^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,v(\tau),{}^H D_a^{\alpha,\beta;\psi}v(\tau)\right)\right|\mathrm{d}\tau\\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}c_2(t,\tau)\left|\mathfrak{u}(\tau)-v(\tau)\right|_1\mathrm{d}\tau\\ &\quad\le \left|\mathfrak{u}-v\right|_E\frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}c_2(t,\tau)E_\alpha\left(\xi(\psi(t)-\psi(a))^\alpha\right)\\ &\quad\le \left|\mathfrak{u}-v\right|_E\lambda_2 E_\alpha\left(\xi(\psi(t)-\psi(a))^\alpha\right).\end{aligned}$$

(20)

From (19) and (20), we have

$$\left|T\mathfrak{u}-Tv\right|_s \le (\lambda_1+\lambda_2)\left|\mathfrak{u}-v\right|_E.$$

Since $\lambda_1 + \lambda_2 < 1$ from Banach fixed theorem gives that T has a unique fixed point in B, the fixed point is also a solution of (1).

Now, our next theorem gives the uniqueness of solutions of (1) without existence part:

Theorem 3.2

Assume that function $\mathfrak{g}$ in (1) and its ψ-Hilfer fractional derivative with respect to t satisfy the condition

$$\left|\mathfrak{g}\left(t,\tau,q_1,q_2\right)-\mathfrak{g}\left(t,\tau,\overline{q}_1,\overline{q_2}\right)\right|\leq\rho\left(t\right)d_1\left(\tau\right)\left[\left|q_1-\overline{q_1}\right|+\left|q_2-\overline{q_2}\right|\right], \tag{21}$$

$$\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,q_1,q_2\right)-{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,\overline{q}_1,\overline{q_2}\right)\right|\leq\rho\left(t\right)d_2\left(\tau\right)\left[\left|q_1-\overline{q_1}\right|+\left|q_2-\overline{q_2}\right|\right], \tag{22}$$

where $\rho, d_1, d_2 \in c\left(\Delta, \mathbb{R}_+\right)$, $d(t) = \max(d_1 t, d_2 t)$ and $p(t) = \rho(t)d(t)$. Then, Eq. (1) has at most one solution on Δ.

Proof

Let $u(t)$ and $v(t)$ be two solutions of Eq. (1), then we have

$$\begin{aligned}
&\left|\mathfrak{u}(t)-v(t)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(t)-{}^{H}D_{a}^{\alpha,\beta;\psi}v(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)\left(\left(\psi(b)-\psi(\tau)\right)\right)\left|\mathfrak{g}\left(t,\tau,\mathfrak{u}(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right)\right. \\
&\quad \left.-\mathfrak{g}\left(t,\tau,v(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}v(\tau)\right)\right|\mathrm{d}\tau \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)\left(\left(\psi(b)-\psi(\tau)\right)\right)\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,\mathfrak{u}(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right)\right. \\
&\quad \left.-{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,v(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}v(\tau)\right)\right|\mathrm{d}\tau \\
&\leq \rho(t)\,d(\tau)\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)\left(\left(\psi(b)-\psi(\tau)\right)^{\alpha-1}\right) \\
&\quad \times\left[\left|\mathfrak{u}(\tau)-v(\tau)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)-{}^{H}D_{a}^{\alpha,\beta;\psi}v(\tau)\right|\right] \\
&=\frac{p(t)}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)\left(\left(\psi(b)-\psi(\tau)\right)^{\alpha-1}\right) \\
&\quad \times\left[\left|\mathfrak{u}(\tau)-v(\tau)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)-{}^{H}D_{a}^{\alpha,\beta;\psi}v(\tau)\right|\right].
\end{aligned}$$

(23)

If we put $(v(t)=0)$ in Lemma 2.1 and applying then

$$\left|\mathfrak{u}(t)-v(t)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(t)-{}^{H}D_{a}^{\alpha,\beta;\psi}v(t)\right|\leq 0,$$

(24)

we get $\mathfrak{u}(t)=v(t)$ which proves the uniqueness of solution of Eq. (1) on Δ.

4 Estimates on solution

Now in our next result, we obtain the estimates on the solutions of Eq. (1) with given suitable conditions on the involved functions.

Theorem 4.1

Assume that the functions $\mathfrak{w}$, $\mathfrak{g}$ in Eq. (1) and its ψ-Hilfer fractional derivative with respect to t satisfy the conditions

$$\left|\mathfrak{w}\left(t\right)\right| + \left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}\left(t\right)\right| \le m(t),$$

(25)

$$\left|\mathfrak{g}\left(t,\tau,q_1,q_2\right)\right| \le \rho\left(t\right) d_1\left(\tau\right)\left[\left|q_1\right| + \left|q_2\right|\right],$$

(26)

$$\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,q_1,q_2\right)\right| \le \rho\left(t\right) d_1\left(\tau\right)\left[\left|q_1\right| + \left|q_2\right|\right],$$

(27)

where $m(t), \rho(t), d_1(\tau), d_2(\tau) \in C(\Delta, \mathbb{R}_+)$. If $\mathfrak{u}(t), t \in \Delta$ *is any solution of Eq. (1), then*

$$\left|\mathfrak{u}\left(t\right)\right| + \left|{}^{H}D_{a}^{\alpha,\beta;\psi}u\left(t\right)\right| \le m(t) + \frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\sum_{k=1}^{\infty}\frac{\left[p(t)\Gamma\left(\alpha\right)\right]^{k}}{\Gamma\left(\alpha k\right)}\psi'\left(\tau\right)\left|\psi\left(b\right)-\psi\left(\tau\right)\right|^{\alpha k-1}m\left(t\right)\mathrm{d}\tau.$$

(28)

Proof

Since $\mathfrak{u}(t), t \in \Delta$ is a solution of Eq. (1), we have

$$\begin{aligned}
\left|\mathfrak{u}\left(t\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(t\right)\right| &\le \left|\mathfrak{w}\left(t\right)\right|+\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\left|\mathfrak{g}\left(t,\tau,\mathfrak{u}\left(t\right),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(t\right)\right)\right|\mathrm{d}\tau\\
&\quad+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}\left(t\right)\right|+\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,\mathfrak{u}\left(t\right),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(t\right)\right)\right|\mathrm{d}\tau\\
&\le m(t)+\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\rho\left(t\right)d_{1}\left(\tau\right)\left[\left|\mathfrak{u}\left(\tau\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}u\left(\tau\right)\right|\right]\mathrm{d}\tau\\
&\quad+\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\rho\left(t\right)d_{2}\left(\tau\right)\left[\left|\mathfrak{u}\left(\tau\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(\tau\right)\right|\right]\mathrm{d}\tau\\
&\le m(t)+\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\rho\left(t\right)d_{1}\left(\tau\right)\left[\left|\mathfrak{u}\left(\tau\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(\tau\right)\right|\right]\mathrm{d}\tau\\
&\quad+\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\rho\left(t\right)d_{2}\left(\tau\right)\left[\left|\mathfrak{u}\left(\tau\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(\tau\right)\right|\right]\mathrm{d}\tau\\
&\le m(t)+\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\rho\left(t\right)\left[d_{1}\left(\tau\right)+d_{2}\left(\tau\right)\right]\left[\left|\mathfrak{u}\left(\tau\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(\tau\right)\right|\right]\mathrm{d}\tau\\
&\le m(t)+\frac{p\left(t\right)}{\Gamma\left(\alpha\right)}\int_{a}^{b}\psi'\left(\tau\right)\left(\psi\left(b\right)-\psi\left(\tau\right)\right)^{\alpha-1}\\
&\quad\times\left[\left|\mathfrak{u}\left(\tau\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}\left(\tau\right)\right|\right]\mathrm{d}\tau
\end{aligned}$$

(29)

Now applying Lemma (2.1) to (29), we get (28).

Remark 1

The estimates obtained in (31) yield not only the bounds on the solution and its fractional derivatives. If the bounds on (31) are bounded, then the solution of Eq. (1) and its fractional derivative are bounded.

Now in our next theorem, we obtain the estimates on the solution of Eq. (1) and here we assume that the function $\mathfrak{g}$ and its ψ-Hilfer fractional derivative with respect to t satisfy the Lipschitz-type conditions.

Theorem 4.2

Assume that hypothesis in Theorem 3.2holds, and let

$$a(t)=\frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\times\left|\mathfrak{g}\left(t,\tau,w(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\right|\mathrm{d}\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\times\left|{}^H D_a^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,\mathfrak{w}(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\right|\mathrm{d}\tau,$$

(30)

for $t\in\Delta$. If $\mathfrak{u}(t), t\in\Delta$ *is a solution of* (1), *then*

$$\left|\mathfrak{u}(t)-\mathfrak{w}(t)\right|+\left|{}^H D_a^{\alpha,\beta;\psi}\mathfrak{u}(t)-{}^H D_a^{\alpha,\beta;\psi}\mathfrak{w}(t)\right|$$
$$\leq a(t)+\int_a^b\sum_{k=1}^{\infty}\frac{[p(t)\,\Gamma(\alpha)]}{\Gamma(\alpha k)}\psi'(\tau)\left[\psi(b)-\psi(\tau)\right]^{\alpha k-1}a(\tau)\,\mathrm{d}\tau.$$

(31)

Proof

As $\mathfrak{u}(t)$ is a solution of (1) and from the hypotheses, we get

$$\begin{aligned} \left|\mathfrak{u}(t)-\mathfrak{w}(t)\right| \le & \frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|\mathfrak{g}\left(t,\tau,\mathfrak{u}(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right)\right. \\ & \left. -\mathfrak{g}\left(t,\tau,\mathfrak{w}(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\right|\mathrm{d}\tau \\ & +\frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|\mathfrak{g}\left(t,\tau,\mathfrak{w}(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\right|\mathrm{d}\tau \\ \le & \frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|\mathfrak{g}\left(t,\tau,w(\tau),{}^H D_a^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\right|\mathrm{d}\tau \\ & +\frac{1}{\Gamma(\alpha)}\int_a^b \psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\rho(t)\,d_1(\tau)\,\mathrm{d}\tau, \end{aligned}$$

(32)

and

$$\begin{aligned}
&\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(t)-{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(t)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\Big|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,\mathfrak{u}(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)\right) \\
&\quad -{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,\mathfrak{w}(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\Big|\,\mathrm{d}\tau \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|{}^{H}D_{a}^{\alpha,\beta;\psi}g\left(t,\tau,w(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\right|\mathrm{d}\tau \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\left|{}^{H}D_{a}^{\alpha,\beta;\psi}g\left(t,\tau,w(\tau),{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right)\right|\mathrm{d}\tau \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1}\rho(t)\,d_{2}(\tau) \\
&\quad \times\left[|\mathfrak{u}(t)-\mathfrak{w}(t)|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)-{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right|\right]\mathrm{d}\tau
\end{aligned} \tag{33}$$

From (32), (33) and (30), we have

$$\begin{aligned}
&|\mathfrak{u}(t)-\mathfrak{w}(t)|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(t)-{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(t)\right| \\
&\leq a(t)+\frac{p(t)}{\Gamma(\alpha)}\int_{a}^{b}\psi'(\tau)(\psi(b)-\psi(\tau))^{\alpha-1} \\
&\quad \times\left[|\mathfrak{u}(t)-\mathfrak{w}(t)|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{u}(\tau)-{}^{H}D_{a}^{\alpha,\beta;\psi}\mathfrak{w}(\tau)\right|\right]\mathrm{d}\tau.
\end{aligned} \tag{34}$$

Now, an application of Lemma (2.1) to (34) gives (31).

5 Continuous dependence

Now in this section, we give the continuous dependence of solution of Eq. (1) and functions involved. Consider Eq. (1) and corresponding ψ-Hilfer fractional integral equation

$$v(t) = W(t) + \int_a^b G\left(t, \tau, v(\tau), {}^H D_a^{\alpha,\beta;\psi} v(\tau)\right) \mathrm{d}\tau$$

(35)

for $-\infty < a \leq t \leq b < \infty$. The functions W, G are continuous and ψ-Hilfer fractional differentiable with respect to t.

Theorem 5.1

Assume that hypotheses of Theorem 4.1hold. Suppose that

$$|\mathfrak{w}(t) - W(t)| + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} \left| \mathfrak{g}\left(t, \tau, v(\tau), {}^H D_a^{\alpha,\beta;\psi} v(\tau)\right) - G\left(t, \tau, v(\tau), {}^H D_a^{\alpha,\beta;\psi} v(\tau)\right) \right| \mathrm{d}\tau \leq h_1(t),$$

(36)

$$\left| {}^H D_a^{\alpha,\beta;\psi} \mathfrak{w}(t) - {}^H D_a^{\alpha,\beta;\psi} W(t) \right| + \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha-1} \left| {}^H D_a^{\alpha,\beta;\psi} g\left(t, \tau, v(\tau), {}^H D_a^{\alpha,\beta;\psi} v(\tau)\right) - {}^H D_a^{\alpha,\beta;\psi} G\left(t, \tau, v(\tau), {}^H D_a^{\alpha,\beta;\psi} v(\tau)\right) \right| \mathrm{d}\tau \leq h_2(t),$$

(37)

where $\mathfrak{w}$, $\mathfrak{g}$ and W, G are the functions involved in (1) and (35) and $h_1(t), h_2(t) \in C(\Delta, \mathbb{R}_+)$.

Let $\mathfrak{u}(t), v(t)$ for $t \in \Delta$ *be the solution of Eqs. (1) and (35), respectively. Then, the solution* $\mathfrak{u}(t), t \in \Delta$ *of (35) depends continuously on functions involved.*

Proof

Since $\mathfrak{u}(t)$ and $v(t)$ are solutions of (1) and (35) and the given hypotheses, we have

$$\begin{aligned}\left|\mathfrak{u}\left(t\right)-v\left(t\right)\right| &\le \left|\mathfrak{w}\left(t\right)-W\left(t\right)\right| \\ &\quad+\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^b\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}\left|\mathfrak{g}\left(t,\tau,v\left(\tau\right),{}^{H}D_a^{\alpha,\beta;\psi}v\left(\tau\right)\right)\right. \\ &\quad\left.-G\left(t,\tau,v\left(\tau\right),{}^{H}D_a^{\alpha,\beta;\psi}v\left(\tau\right)\right)\right|\mathrm{d}\tau \\ &\le h_1\left(t\right)+\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^b\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}\rho\left(t\right)d_1\left(\tau\right) \\ &\quad\times\left[\left|\mathfrak{u}\left(t\right)-v\left(t\right)\right|+\left|{}^{H}D_a^{\alpha,\beta;\psi}\mathfrak{u}\left(t\right)-{}^{H}D_a^{\alpha,\beta;\psi}v\left(t\right)\right|\right]\mathrm{d}\tau,\end{aligned}$$

(38)

and

$$\begin{aligned}&\left|{}^{H}D_a^{\alpha,\beta;\psi}\mathfrak{u}\left(t\right)-{}^{H}D_a^{\alpha,\beta;\psi}v\left(t\right)\right| \\ &\quad\le\left|{}^{H}D_a^{\alpha,\beta;\psi}\mathfrak{w}\left(t\right)-{}^{H}D_a^{\alpha,\beta;\psi}W\left(t\right)\right| \\ &\qquad+\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^b\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}\left|{}^{H}D_a^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,v\left(\tau\right),{}^{H}D_a^{\alpha,\beta;\psi}v\left(\tau\right)\right)\right. \\ &\qquad\left.-{}^{H}D_a^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,v\left(\tau\right),{}^{H}D_a^{\alpha,\beta;\psi}v\left(\tau\right)\right)\right|\mathrm{d}\tau \\ &\qquad+\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^b\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}\left|{}^{H}D_a^{\alpha,\beta;\psi}\mathfrak{g}\left(t,\tau,v\left(\tau\right),{}^{H}D_a^{\alpha,\beta;\psi}v\left(\tau\right)\right)\right. \\ &\qquad\left.-{}^{H}D_a^{\alpha,\beta;\psi}G\left(t,\tau,v\left(\tau\right),{}^{H}D_a^{\alpha,\beta;\psi}v\left(\tau\right)\right)\right|\mathrm{d}\tau \\ &\quad\le h_2\left(t\right)+\frac{1}{\Gamma\left(\alpha\right)}\int\limits_a^b\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}\rho\left(t\right)d_2\left(\tau\right) \\ &\qquad\times\left[\left|\mathfrak{u}\left(t\right)-v\left(t\right)\right|+\left|{}^{H}D_a^{\alpha,\beta;\psi}\mathfrak{u}\left(t\right)-{}^{H}D_a^{\alpha,\beta;\psi}v\left(t\right)\right|\right]\mathrm{d}\tau.\end{aligned}$$

(39)

Thus from (38) and (39), we have

$$\begin{aligned}&\left|\mathrm{u}\left(t\right)-v\left(t\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathrm{u}\left(t\right)-{}^{H}D_{a}^{\alpha,\beta;\psi}v\left(t\right)\right|\\&\quad\le\left(h_{1}\left(t\right)+h_{2}\left(t\right)\right)+\frac{p\left(t\right)}{\Gamma\left(\alpha\right)}\int\limits_{a}^{b}\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha-1}\\&\qquad\times\left[\left|\mathrm{u}\left(t\right)-v\left(t\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathrm{u}\left(t\right)-{}^{H}D_{a}^{\alpha,\beta;\psi}v\left(t\right)\right|\right]\mathrm{d}\tau.\end{aligned}$$

(40)

Now, an application of Lemma (2.1) to (40) gives

$$\begin{aligned}&\left|\mathrm{u}\left(t\right)-v\left(t\right)\right|+\left|{}^{H}D_{a}^{\alpha,\beta;\psi}\mathrm{u}\left(t\right)-{}^{H}D_{a}^{\alpha,\beta;\psi}v\left(t\right)\right|\\&\quad\le\left(h_{1}\left(t\right)+h_{2}\left(t\right)\right)\\&\qquad+\int\limits_{a}^{b}\sum_{k=1}^{\infty}\frac{[p(t)\Gamma\left(\alpha\right)]}{\Gamma\left(\alpha k\right)}\psi'\left(\tau\right)(\psi\left(b\right)-\psi\left(\tau\right))^{\alpha k-1}\left(h_{1}\left(\tau\right)+h_{2}\left(\tau\right)\right)\mathrm{d}\tau.\end{aligned}$$

(41)

Thus, Eq. (41) implies that solution of Eq. (1) depends continuously on the functions involved on the right side of Eq. (1).

Remark 2

If we substitute values for $\psi(x)$ as x, lnx, x^{σ}, then the above equations reduce to of Fredholm type of fractional differential equations such as Riemann Liouville, Hadamard and Erdelyi–Kober fractional differential equations, respectively.

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