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Existence and Ulam stability results of a coupled system for terminal value problems involving ψ-Hilfer fractional operator

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1 Introduction

Over the past few years, mathematicians have realized that fractional calculus has many applications in various scopes of applied science and engineering. Several researchers have employed the fractional calculus as an experiential style of describing the properties of natural phenomena such as chemistry, biology, physics, bioengineering, electrochemistry, finance, economic, etc., for more details, see [12, 20, 21, 23] and many other references. The interesting issue about this theme is that it is completely unlike classical derivatives, because it deals with arbitrary and noninteger order, e.g., the fractional derivative of noninteger order depends not just on the diagram of the function very near to the point but also on some chronicle. Recently, there has been considerable growth in fractional differential equations (FDEs) involving several different fractional derivative operators, we indicate here the more famous operators like Riemann–Liouville (RL), Caputo, Hilfer, Hadamard, Katugampola, and several other generalized operators.
So, this implies that different categories of FDEs involving several fractional operators have been considered. Researchers who are concerned with this topic have presented many generalizations of fractional derivatives like Hilfer–Katugampola, Hilfer–Hadamard, ψ-Caputo,

short, the existence results for ordinary and fractional TVPs have been studied by many investigators, see [5, 13–15, 27, 28, 35]. For example, Benchohra et al. in [13] obtained the existence and uniqueness of solution to the fractional implicit TVP

$$ {}^{\rho}D^{\theta,\eta}_{a^+}y(t) = f\bigl(t, y(t), {}^{\rho}D^{\theta,\eta}_{a^+}y(t)\bigr), \quad a < t \leq T, a > 0, $$

(1)

under the terminal condition

$$ y(T) = w \in \mathbb{R}, $$

(2)

where ${}^{\rho}D^{\theta,\eta}_{a^+}$ is the fractional derivative of order (θ, η) in the Hilfer–Katugampola sense ($0 < \theta < 1, 0 \leq \eta \leq 1$), $\rho > 0$, and $f : (a, T] \times \mathbb{R} \to \mathbb{R}$ is a certain function. Motivated by the aforementioned works, the target of this work is to investigate the existence, uniqueness, and Ulam–Hyers stability of solutions of a coupled system for fractional TVPs involving generalized Hilfer fractional derivative of the type

$$ \begin{cases} D^{\theta_1,\eta_1;\psi}_{a^+}y(t) = f_1(t, x(t)), & a < t \leq T, a > 0, \\ D^{\theta_2,\eta_2;\psi}_{a^+}x(t) = f_2(t, y(t)), & a < t \leq T, a > 0 \end{cases} $$

Hilfer fractional derivative of order θ_i and type η_i with respect to ψ and $f : (a, T] \times \mathbb{R} \to \mathbb{R}$ is a certain function under the conditions listed later.

As far as we know, no papers about a coupled system for fractional TVPs exist in the literature, specifically for those encompassing the generalized fractional derivative in the ψ-Hilfer sense. Moreover, the results of the problem at hand are obtained under minimal assumptions on nonlinear functions f_1, f_2.

The rest of the structure of this paper is as follows. In Sect. 2, we briefly state some essential definitions and the results that are applied throughout the paper. Section 3 studies the existence and uniqueness results on ψ-Hilfer FDEs with the terminal conditions via fixed point techniques of Banach and Krasnoselskii. The stability analysis in the concept Ulam–Hyers of the proposed system is investigated in Sect. 4. At the end, some examples are included to illustrate the applicability of the obtained results in Sect. 5.

2 Auxiliary results

Let $[a, T] \subset \mathbb{R}^+$ with $(0 < a < T < \infty)$, we also consider $C[a, T]$ the Banach space of real-valued continuous functions defined on $[a, T]$ with the norm

Obviously, $C_{1-\varsigma;\psi}[a,T]$ and $C^{n}_{1-\varsigma;\psi}[a,T]$ are Banach spaces endowed with the norms

$$\|\sigma\|_{C_{1-\varsigma;\psi}} = \max_{t\in[a,T]}\left|\left[\psi(t)-\psi(a)\right]^{1-\varsigma}\sigma(t)\right|,$$
$$\|\sigma\|_{C^{n}_{1-\varsigma;\psi}} = \sum_{j=0}^{n-1}\left\|\sigma^{(j)}\right\|_{C} + \left\|\sigma^{(n)}\right\|_{C_{1-\varsigma;\psi}},$$

respectively. For $n=0$, $C^{0}_{1-\varsigma;\psi}[a,T]=C_{1-\varsigma;\psi}[a,T]$. Let us introduce the following space:

$$E = \left\{\sigma(t) : \sigma(t)\in C_{1-\varsigma;\psi}[a,T]\right\}$$

endowed with the norm defined by

$$\|\sigma\|_{E} = \|\sigma\|_{c_{1-\varsigma;\psi}}.$$

It is easy to perceive that, for $\sigma\in E$, $(E,\|\sigma\|_{E})$ is a Banach space. Then, for $(\sigma,\rho)\in E\times E$, the product space $(E\times E,\|(\sigma,\rho)\|_{E\times E})$ is a Banach space too, where

$$\left\|(\sigma,\rho)\right\|_{E\times E} = \max\left(\|\sigma\|_{E},\|\rho\|_{E}\right), \quad \sigma,\rho\in E.$$

Definition 1

([19, Sect. 2.5, Eq. (2.5.1)])

$$D_{a^+}^{\theta,\psi}\sigma(t) = D^{n,\psi} I_{a^+}^{n-\theta,\psi}\sigma(t)$$

and

$${}^{C}D_{a^+}^{\theta,\psi}\sigma(t) = I_{a^+}^{n-\theta,\psi}\sigma_{\psi}^{[n]}(t),$$

respectively, where $\sigma_{\psi}^{[n]}(t) = D^{n,\psi}\sigma(t)$, $D^{n,\psi} = [\frac{1}{\psi'(t)}\frac{d}{dt}]^n$, and $n = [\theta] + 1$.

Definition 3

([29, Definition 7])
Let $\varsigma = \theta + \eta(n-\theta)$ where $n-1 < \theta < n \in \mathbb{N}$, $0 \leq \eta \leq 1$, and $\sigma \in C^n[a,T]$. Then the left-sided ψ-fractional derivative in the concept ψ-Hilfer of order θ and type η of a function σ w.r.t. ψ is given by

$$\begin{aligned} D_{a^+}^{\theta,\eta,\psi}\sigma(t) &= I_{a^+}^{\eta(n-\theta);\psi} D^{n,\psi} I_{a^+}^{(1-\eta)(n-\theta);\psi}\sigma(t) \\ &= I_{a^+}^{\eta(n-\theta);\psi} D_{a^+}^{\varsigma;\psi}\sigma(t), \end{aligned}$$

(5)

where

$$D_{a^+}^{\varsigma;\psi}\sigma(t) = D^{n,\psi} I_{a^+}^{(1-\eta)(n-\theta);\psi}\sigma(t).$$

$$I_{a^+}^{\theta;\psi}\sigma(a) = \lim_{t\to a^+} I_{a^+}^{\theta;\psi}\sigma(t) = 0.$$

Definition 4

[32, Definition 3] The weighted continuous spaces $C_{1-\varsigma;\psi}^{\theta,\eta}[a,T]$ and $C_{1-\varsigma;\psi}^{\varsigma}[a,T]$ are described by

$$C_{1-\varsigma;\psi}^{\theta,\eta}[a,T] = \{\sigma \in C_{1-\varsigma;\psi}[a,T], D_{a^+}^{\theta,\eta;\psi}\sigma \in C_{1-\varsigma;\psi}[a,T]\}$$

and

$$C_{1-\varsigma;\psi}^{\varsigma}[a,T] = \{\sigma \in C_{1-\varsigma;\psi}[a,T], D_{a^+}^{\varsigma;\psi}\sigma \in C_{1-\varsigma;\psi}[a,T]\},$$

(6)

where $\varsigma = \theta + \eta(1-\theta)$, $0 < \theta < 1$, and $0 \le \eta \le 1$.
Observe that

$$C_{1-\varsigma;\psi}^{\varsigma}[a,T] \subset C_{1-\varsigma;\psi}^{\theta,\eta}[a,T] \subset C_{1-\varsigma;\psi}[a,T] \subset C[a,T].$$

Lemma 3

([2, Lemma 2.3])
Let $\varsigma = \theta + \eta(1-\theta)$ *where* $0 < \theta < 1, 0 \le \eta \le 1$, *and* $\sigma \in C_{1-\varsigma;\psi}^{\varsigma}[a,T]$. *Then*

$I_{a^+}^{1-\varsigma;\psi}\sigma \in C_{1-\varsigma,\psi}^1[a,T]$. *Then*

$$I_{a^+}^{\varsigma;\psi} D_{a^+}^{\varsigma,\psi}\sigma(t) = \sigma(t) - \frac{I_{a^+}^{1-\varsigma;\psi}\sigma(a)}{\Gamma(\varsigma)}\big(\psi(t)-\psi(a)\big)^{\varsigma-1}.$$

Lemma 6

([19, Property 2.18])

Let $t > a$, *and consider* $\chi^\varsigma(t) := [\psi(t)-\psi(a)]^{\varsigma-1}$. *Then, for* $\theta > 0$ *and* $\varsigma > 0$,

$$I_{a^+}^{\theta,\psi}\chi^\varsigma(t) = \frac{\Gamma(\varsigma)}{\Gamma(\theta+\varsigma)}\big(\psi(t)-\psi(a)\big)^{\theta+\varsigma-1}.$$

Besides, for $0 < \theta < 1$,

$$D_{a^+}^{\theta,\psi}\chi^\theta(t) = 0.$$

Theorem 1

([36, Theorem 1.45])

Let Ξ *be a Banach space, and let* $\varnothing \subset \Xi$ *be a nonempty, closed, convex, and bounded set, and* Π_1, Π_2 *be two operators satisfying*

(i) $\Pi_1 u + \Pi_2 v \in \varnothing$ *for all* $u, v \in \varnothing$;
(ii) Π_1 *is continuous and compact*;
(iii) Π_2 *is a contraction operator*.

3 Existence and uniqueness results

To shorten the length of equations, we set
$\mathcal{K}_{\psi}^{\varsigma}(t,a) := [\psi(t)-\psi(a)]^{\varsigma-1}$ and
$\mathcal{H}_{\psi}^{\theta}(t,s) := \psi'(s)[\psi(t)-\psi(s)]^{\theta-1}$.

Theorem 3

Let $\varsigma = \theta + \eta(1-\theta)$ *where* $0 < \theta < 1$ *and* $0 \leq \eta \leq 1$. *If* $\sigma : (a,T] \to \mathbb{R}$ *is a function such that* $\sigma(\cdot) \in C_{1-\varsigma,\psi}[a,T]$, *then* $y \in C_{1-\varsigma,\psi}^{\varsigma}(a,T]$ *satisfies the TVP for* ψ*-Hilfer FDEs*

$$D_{a^+}^{\theta,\eta;\psi} y(t) = \sigma(t), \quad t \in (a,T], a > 0,$$

(7)

$$y(T) = w \in \mathbb{R},$$

(8)

if and only if y *fulfills the following fractional integral equation*:

$$y(t) = \frac{\mathcal{K}_{\psi}^{\varsigma}(t,a)}{\mathcal{K}_{\psi}^{\varsigma}(T,a)}\left[w - \frac{1}{\Gamma(\theta)}\int_a^T \mathcal{H}_{\psi}^{\theta}(T,s)\sigma(s)\,ds\right] + \frac{1}{\Gamma(\theta)}\int_a^t \mathcal{H}_{\psi}^{\theta}(t,s)\sigma(s)\,ds.$$

(9)

$$I_{a^+}^{1-\varsigma,\psi}y(t)\in C_{1-\varsigma,\psi}^{1}[a,T].$$

(11)

Take advantage of Lemma 5 to get

$$I_{a^+}^{\varsigma;\psi}D_{a^+}^{\varsigma,\psi}y(t)=y(t)-\frac{I_{a^+}^{1-\varsigma;\psi}y(a)}{\Gamma(\varsigma)}\mathcal{K}_{\psi}^{\varsigma}(t,a),\quad t\in(a,T].$$

(12)

It follows from the assumption $y\in C_{1-\varsigma,\psi}^{\varsigma}[a,T]$, Lemma 3, and equation (7) that

$$I_{a^+}^{\varsigma;\psi}D_{a^+}^{\varsigma;\psi}y(t)=I_{a^+}^{\theta;\psi}\sigma(t).$$

(13)

Equating both sides of equations (12) and (13), we find that

$$y(t)=\frac{I_{a^+}^{1-\varsigma;\psi}y(a)}{\Gamma(\varsigma)}\mathcal{K}_{\psi}^{\varsigma}(t,a)+I_{a^+}^{\theta;\psi}\sigma(t).$$

(14)

Using the terminal condition $y(T)=w$, we get

Conversely, suppose that $y \in C^{\varsigma}_{1-\varsigma,\psi}[a,T]$ satisfying integral equation (9). Applying fractional derivative $D^{\varsigma;\psi}_{a^+}$ on both sides of integral equation (9) and employing Lemmas 6 and 3, we will surely find

$$\begin{aligned} D^{\varsigma;\psi}_{a^+} y(t) =& \frac{1}{\mathcal{K}^{\varsigma}_{\psi}(T,a)} \left[w - I^{\theta;\psi}_{a^+} \sigma(T) \right] D^{\varsigma;\psi}_{a^+} \mathcal{K}^{\varsigma}_{\psi}(t,a) + D^{\varsigma;\psi}_{a^+} I^{\theta;\psi}_{a^+} \sigma(t) \\ =& D^{\eta(1-\theta);\psi}_{a^+} \sigma(t), \end{aligned}$$

(16)

where $D^{\varsigma;\psi}_{a^+} \mathcal{K}^{\varsigma}_{\psi}(t,a) = 0$. From (10), we have $D^{\varsigma;\psi}_{a^+} y \in C_{1-\varsigma;\psi}[a,T]$, then (16) implies

$$D^{\varsigma;\psi}_{a^+} y(t) = D^{1,\psi} I^{1-\eta(1-\theta),\psi}_{a^+} \sigma(t) = D^{\eta(1-\theta);\psi}_{a^+} \sigma(t) \in C_{1-\varsigma;\psi}[a,T].$$

(17)

As $\sigma(t) \in C_{1-\varsigma;\psi}[a,T]$, it follows from Lemma 1 that

$$I^{1-\eta(1-\theta);\psi}_{a^+} \sigma \in C_{1-\varsigma;\psi}[a,T].$$

(18)

From the definition of $C^{n}_{1-\varsigma;\psi}(a,T]$ with the aid of equations (17), (18), we get

From (5) with $n = 1$, equation (19) reduces to

$$D_{a^+}^{\theta,\eta;\psi} y(t) = \sigma(t), \quad t \in (a, T].$$

This proves that y also satisfies FDE (7). Undoubtedly, if $y \in C_{1-\varsigma;\psi}^{\varsigma}[a, T]$ satisfies integral equation (9), then it also fulfills terminal condition (8). □

Before we present our main results, we consider that the following assumptions are satisfied:

(H₁):

$f_1, f_2 : (a, T] \times \mathbb{R} \to \mathbb{R}$ are such that

$$f_1(\cdot, y(\cdot)) \in C_{1-\varsigma_1;\psi}^{\eta_1(1-\theta_1)}[a, T], \quad y \in C_{1-\varsigma_1;\psi}[a, T],$$
$$f_2(\cdot, x(\cdot)) \in C_{1-\varsigma_2;\psi}^{\eta_2(1-\theta_2)}[a, T], \quad y \in C_{1-\varsigma_2;\psi}[a, T].$$

(H₂):

There exist $L_1(> 0)$ and $L_2(> 0)$ such that, for $p, p^*, q, q^* \in \mathbb{R}$, $t \in (a, T]$, we have

$$\left|f_1(t, p) - f_1(t, p^*)\right| \le L_1\left|p - p^*\right|,$$
$$\left|f_2(t, q) - f_2(t, q^*)\right| \le L_2\left|q - q^*\right|.$$

In the forthcoming theorem, by using Theorem 2, we prove the unique solution of a coupled system for ψ-Hilfer terminal FDEs (3)–(4). In view of Theorem 3, we get the

$$\begin{cases} y(t) = \frac{\mathcal{K}_{\psi}^{\varsigma_1}(t,a)}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)}[w_1 - \frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_{\psi}^{\theta_1}(T,s)f_1(s,x(s))\,ds] \\ \qquad + \frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_{\psi}^{\theta_1}(t,s)f_1(s,x(s))\,ds, \\ x(t) = \frac{\mathcal{K}_{\psi}^{\varsigma_2}(t,a)}{\mathcal{K}_{\psi}^{\varsigma_2}(T,a)}[w_2 - \frac{1}{\Gamma(\theta_2)}\int_a^T \mathcal{H}_{\psi}^{\theta_2}(T,s)f_2(s,y(s))\,ds] \\ \qquad + \frac{1}{\Gamma(\theta_2)}\int_a^t \mathcal{H}_{\psi}^{\theta_2}(t,s)f_2(s,y(s))\,ds. \end{cases}$$

(20)

According to Lemma (7), we consider the operators $\mathcal{N}_1 : E \to E$ and $\mathcal{N}_2 : E \to E$ defined by

$$\begin{cases} \mathcal{N}_1 x(t) = y(t), \\ \mathcal{N}_2 y(t) = x(t). \end{cases}$$

That is,

$$\begin{cases} \mathcal{N}_1 x(t) = \frac{\mathcal{K}_{\psi}^{\varsigma_1}(t,a)}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)}[w_1 - \frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_{\psi}^{\theta_1}(T,s)f_1(s,x(s))\,ds] \\ \qquad + \frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_{\psi}^{\theta_1}(t,s)f_1(s,x(s))\,ds, \\ \mathcal{N}_2 y(t) = \frac{\mathcal{K}_{\psi}^{\varsigma_2}(t,a)}{\mathcal{K}_{\psi}^{\varsigma_2}(T,a)}[w_2 - \frac{1}{\Gamma(\theta_2)}\int_a^T \mathcal{H}_{\psi}^{\theta_2}(T,s)f_2(s,y(s))\,ds] \\ \qquad + \frac{1}{\Gamma(\theta_2)}\int_a^t \mathcal{H}_{\psi}^{\theta_2}(t,s)f_2(s,y(s))\,ds. \end{cases}$$

(21)

$$\begin{aligned}
\Lambda_{f_1} &= \left[\frac{2L_1\mathcal{B}(\varsigma_1,\theta_1)}{\Gamma(\theta_1)}\right]\mathcal{K}_{\psi}^{\theta_1+1}(T,a),\\
\Delta_{f_1} &= \frac{w_1}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)} + \frac{2A_1}{\Gamma(\theta_1+1)}\mathcal{K}_{\psi}^{2}(T,a),\\
\Lambda_{f_2} &= \left[\frac{2L_2\mathcal{B}(\varsigma_2,\theta_2)}{\Gamma(\theta_2)}\right]\mathcal{K}_{\psi}^{\theta_2+1}(T,a),\\
\Delta_{f_2} &= \frac{w_2}{\mathcal{K}_{\psi}^{\varsigma_2}(T,a)} + \frac{2A_2}{\Gamma(\theta_2+1)}\mathcal{K}_{\psi}^{2}(T,a),
\end{aligned}$$

where $A_i = \max_{t\in[a,T]}|f_i(t,0)|$, $i=1,2$.
Now, via Theorems 2, 1, we obtain the existence and uniqueness results of a coupled system for ψ-Hilfer FDEs (3)–(4).

Theorem 4

Assume that (H_1) *and* (H_2) *hold. If* $\Lambda_{f_1} < 1$ *and* $\Lambda_{f_2} < 1$, *then* ψ*-Hilfer coupled system* (3)–(4) *has a unique solution in* $E^{\varsigma}\times E^{\varsigma}\subset E^{\theta,\eta}\times E^{\theta,\eta}$, *where* $E^{\varsigma} := C^{\varsigma}_{1-\varsigma;\psi}[a,T]$ *and* $E^{\theta,\eta} := C^{\theta,\eta}_{1-\varsigma;\psi}[a,T]$.

Proof

Define the closed, bounded, convex, and nonempty set

$$\mathcal{S}_R = \left\{(y,x)\in E\times E : \left\|(y,x)\right\|_{E\times E}\le R\right\}\subset E\times E,$$

$$\begin{aligned}
&\left|\left[\psi(t)-\psi(a)\right]^{1-\varsigma_1}(\mathcal{N}_1 x)(t)\right| \\
&\quad\leq \frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)}\left[w_1+\frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_{\psi}^{\theta_1}(T,s)\left|f_1\left(s,x(s)\right)\right|\,ds\right] \\
&\qquad+\frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t,a)}\frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_{\psi}^{\theta_1}(t,s)\left|f_1\left(s,x(s)\right)\right|\,ds \\
&\quad\leq \frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)}\left[w_1+\frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_{\psi}^{\theta_1}(T,s)\right. \\
&\qquad\left.\times\left[\left|f_1\left(s,x(s)\right)-f_1(s,0)\right|+\left|f_1(s,0)\right|\right]\,ds\right] \\
&\qquad+\frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t,a)}\frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_{\psi}^{\theta_1}(t,s) \\
&\qquad\times\left[\left|f_1\left(s,x(s)\right)-f_1(s,0)\right|+\left|f_1(s,0)\right|\right]\,ds.
\end{aligned}$$

(22)

Thanks to hypothesis (H_2), for $(y,x)\in\mathcal{S}_R$, $t\in(a,T]$, we have

$$\begin{aligned}
&\left|\left[\psi(t)-\psi(a)\right]^{1-\varsigma_1}(\mathcal{N}_1 x)(t)\right| \\
&\quad\leq \frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)}\left[w_1+\frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_{\psi}^{\theta_1}(T,s)\left(L_1\left(\psi(s)-\psi(a)\right)^{1-\varsigma_1}\|x\|_E+A_1\right)ds\right] \\
&\qquad+\frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t,a)}\frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_{\psi}^{\theta_1}(t,s)\left(L_1\left(\psi(s)-\psi(a)\right)^{\varsigma_1-1}\|x\|_E+A_1\right)ds \\
&\quad\leq \frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)}\left[w_1+L_1R\frac{\Gamma(\varsigma_1)}{\Gamma(\varsigma_1+\theta_1)}\mathcal{K}_{\psi}^{\theta_1+\varsigma_1}(T,a)+\frac{A_1}{\Gamma(\theta_1+1)}\mathcal{K}_{\psi}^{\theta_1+1}(T,a)\right]
\end{aligned}$$

and

$$\mathcal{K}_{\psi}^{\varsigma_1}(t,a) \leq 1 \quad \text{for } 0 < \varsigma_1 \leq 1.$$

Hence, inequality (23) becomes

$$\left|\left[\psi(t)-\psi(a)\right]^{1-\varsigma_1}(\mathcal{N}_1 x)(t)\right| \leq \Lambda_{f_1} + \Delta_{f_1} R,$$

which leads to

$$\|\mathcal{N}_1 x\|_E \leq \Lambda_{f_1} + \Delta_{f_1} R \leq R.$$

(24)

Similarly, we can get that

$$\|\mathcal{N}_2 y\|_E \leq \Lambda_{f_2} + \Delta_{f_2} R \leq R.$$

(25)

It follows from (24) and (25) that

$$\|\mathcal{N}(y,x)\|_{E\times E} = \max\left(\|\mathcal{N}_1 x\|_E, \|\mathcal{N}_2 y\|_E\right) \leq \max(R,R) = R.$$

This proves $\mathcal{N}\mathcal{S}_R \subset \mathcal{S}_R$.
Step(2): The operator $\mathcal{N}$ is a contraction.

$$\begin{aligned}
&\left|[\psi(t)-\psi(a)]^{1-\varsigma_1}\left[(\mathcal{N}_1x)(t)-(\mathcal{N}_1x^*)(t)\right]\right| \\
&\quad\leq \frac{1}{\mathcal{K}_\psi^{\varsigma_1}(T,a)}\frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_\psi^{\theta_1}(T,s)\left|f_1\left(s,x(s)\right)-f_1\left(s,x^*(s)\right)\right|ds \\
&\qquad+\frac{1}{\mathcal{K}_\psi^{\varsigma_1}(t,a)}\frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_\psi^{\theta_1}(t,s)\left|f_1\left(s,x(s)\right)-f_1\left(s,x^*(s)\right)\right|ds \\
&\quad\leq \frac{1}{\mathcal{K}_\psi^{\varsigma_1}(T,a)}L_1\left\|x-x^*\right\|_E\left(I_{a^+}^{\theta_1,\psi}\mathcal{K}_\psi^{\varsigma_1}(s,a)\right)(T) \\
&\qquad+\frac{1}{\mathcal{K}_\psi^{\varsigma_1}(t,a)}L_1\left\|x-x^*\right\|_E\left(I_{a^+}^{\theta_1,\psi}\mathcal{K}_\psi^{\varsigma_1}(s,a)\right)(t) \\
&\quad\leq 2L_1\frac{\mathcal{B}(\varsigma_1,\theta_1)}{\Gamma(\theta_1)}\mathcal{K}_\psi^{\theta_1+1}(T,a)\left\|x-x^*\right\|_E,
\end{aligned}$$

which implies

$$\left\|(\mathcal{N}_1x)-(\mathcal{N}_1x^*)\right\|_E\leq\Lambda_{f_1}\left\|x-x^*\right\|_E.$$

By the same technique, we can also get

$$\left\|(\mathcal{N}_2y)-(\mathcal{N}_2y^*)\right\|_E\leq\Lambda_{f_2}\left\|x-x^*\right\|_E.$$

In view of the conditions $\Lambda_{f_1}<1$ and $\Lambda_{f_2}<1$, we get

$$\left\|\mathcal{N}(y,x)-\mathcal{N}\left(y^*,x^*\right)\right\|_{E\times E}<\left\|(y,x)-\left(y^*,x^*\right)\right\|_{E\times E}.$$

$$\begin{cases} \hat{y}(t) = \frac{\mathcal{K}_{\psi}^{\varsigma_1}(t,a)}{\mathcal{K}_{\psi}^{\varsigma_1}(T,a)}[w_1 - \frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_{\psi}^{\theta_1}(T,s) f_1(s,\widehat{x}(s))\,ds] \\ \qquad + \frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_{\psi}^{\theta_1}(t,s) f_1(s,\widehat{x}(s))\,ds, \\ \widehat{x}(t) = \frac{\mathcal{K}_{\psi}^{\varsigma_2}(t,a)}{\mathcal{K}_{\psi}^{\varsigma_2}(T,a)}[w_2 - \frac{1}{\Gamma(\theta_2)}\int_a^T \mathcal{H}_{\psi}^{\theta_2}(T,s) f_2(s,\hat{y}(s))\,ds] \\ \qquad + \frac{1}{\Gamma(\theta_2)}\int_a^t \mathcal{H}_{\psi}^{\theta_2}(t,s) f_2(s,\hat{y}(s))\,ds. \end{cases}$$

Multiplying both sides of the last system by $D_{a^+}^{\varsigma_1,\psi}$, $D_{a^+}^{\varsigma_2,\psi}$ respectively, it follows from Lemmas 6 and 3 that

$$\begin{cases} D_{a^+}^{\varsigma_1,\psi}\hat{y}(t) = D_{a^+}^{\varsigma_1,\psi} I_{a^+}^{\theta_1;\psi} f_1(s,\widehat{x}(s))(t) = D_{a^+}^{\eta_1(1-\theta_1);\psi} f_1(s,\widehat{x}(s)), \\ D_{a^+}^{\varsigma_2,\psi}\widehat{x}(t) = D_{a^+}^{\varsigma_2,\psi} I_{a^+}^{\theta_2;\psi} f_2(s,\hat{y}(s))(t) = D_{a^+}^{\eta_2(1-\theta_2);\psi} f_2(s,\hat{y}(s)). \end{cases}$$

Since $\varsigma_i \geq \theta_i$ $(i = 1,2)$ and by (H_1), we get

$$\begin{cases} D_{a^+}^{\eta_1(1-\theta_1);\psi} f_1(s,\widehat{x}(s)) \in C_{1-\varsigma_1;\psi}[a,T], \\ D_{a^+}^{\eta_2(1-\theta_2);\psi} f_2(s,\hat{y}(s)) \in C_{1-\varsigma_2;\psi}[a,T]. \end{cases}$$

Hence, $D_{a^+}^{\varsigma_1,\psi}\hat{y} \in C_{1-\varsigma_1;\psi}[a,T]$ and $D_{a^+}^{\varsigma_2,\psi}\widehat{x} \in C_{1-\varsigma_2;\psi}[a,T]$, it follows from the definition of $C_{1-\varsigma_i;\psi}^{\varsigma_i}[a,T]$ $(i = 1,2)$ that $\hat{y} \in C_{1-\varsigma_1;\psi}^{\varsigma_1}[a,T]$ and $\widehat{x} \in C_{1-\varsigma_2;\psi}^{\varsigma_2}[a,T]$. As a sequel to the steps outlined above, we infer that the ψ-Hilfer coupled system (3)–(4) has a unique solution in $C_{1-\varsigma_1;\psi}^{\varsigma_1}[a,T] \times C_{1-\varsigma_2;\psi}^{\varsigma_2}[a,T]$. □

$$
\begin{aligned}
(\mathcal{F}_1 x)(t) &= \frac{1}{\Gamma(\theta_1)} \int_a^t \mathcal{H}_\psi^{\theta_1}(t,s) f_1\big(s, x(s)\big)\, ds, \\
(\mathcal{G}_1 x)(t) &= \frac{\mathcal{K}_\psi^{\varsigma_1}(t,a)}{\mathcal{K}_\psi^{\varsigma_1}(T,a)} \left[w_1 - \frac{1}{\Gamma(\theta_1)} \int_a^T \mathcal{H}_\psi^{\theta_1}(T,s) f_1\big(s, x(s)\big)\, ds \right], \\
(\mathcal{F}_2 y)(t) &= \frac{1}{\Gamma(\theta_2)} \int_a^t \mathcal{H}_\psi^{\theta_2}(t,s) f_2\big(s, y(s)\big)\, ds, \\
(\mathcal{G}_2 y)(t) &= \frac{\mathcal{K}_\psi^{\varsigma_2}(t,a)}{\mathcal{K}_\psi^{\varsigma_2}(T,a)} \left[w_2 - \frac{1}{\Gamma(\theta_2)} \int_a^T \mathcal{H}_\psi^{\theta_2}(T,s) f_2\big(s, y(s)\big)\, ds \right].
\end{aligned}
$$

From the above-mentioned operators, we are able to write $\mathcal{N}_1 = \mathcal{F}_1 + \mathcal{G}_1$ and $\mathcal{N}_2 = \mathcal{F}_2 + \mathcal{G}_2$. Thus, the operator $\mathcal{N}$ can be expressed as

$$
\mathcal{N} = \mathcal{F} + \mathcal{G} \quad \text{such that } \mathcal{F}(y,x) = (\mathcal{F}_1 x, \mathcal{F}_2 y) \text{ and } \mathcal{G}(y,x) = (\mathcal{G}_1 x, \mathcal{G}_2 y).
$$

The proof will be divided into several stages as follows:
Stage(1): $\mathcal{N}$ is continuous.
The continuity of f_1 and f_2 implies the continuity of $\mathcal{N}$.
Stage(2): $\mathcal{F}(\mathcal{K})$ is uniformly bounded.
Let $(y,x) \in \mathcal{K}$, $t \in (a,T]$. Then, by using (H_2), we have

$$
\begin{aligned}
\|\mathcal{F}_1 x\|_E &= \max_{t\in[a,T]} \left| \big[\psi(t) - \psi(a)\big]^{1-\varsigma_1} (\mathcal{F}_1 x)(t) \right| \\
&\leq \max \frac{1}{\quad} \frac{1}{\quad} \int_a^t \mathcal{H}_\psi^{\theta_1}(t,s) f_1(s, x(s))\, ds
\end{aligned}
$$

where $\mathcal{K}_{\psi}^{\theta_1+1}(t,a) < \mathcal{K}_{\psi}^{2}(t,a)$for $0 < \theta_1 < 1$. In the same fashion, we get

$$\|\mathcal{F}_2 y\|_E \leq \left[\frac{A_2}{\Gamma(\theta_2+1)} + L_2 \frac{\mathcal{B}(\varsigma_2,\theta_2)}{\Gamma(\theta_2)}\right]\mathcal{K}_{\psi}^{2}(T,a)\|y\|_E := R_2,$$

which implies

$$\left\|\mathcal{F}(y,x)\right\|_{E\times E} \leq \max(R_1,R_2).$$

This proves $\mathcal{F}(\mathcal{K})$ is uniformly bounded.
Stage(3): $\mathcal{F}(\mathcal{K})$ is equicontinuous in $\mathcal{K}$.
Let $(y,x) \in \mathcal{K}$ and $t_1, t_2 \in (a,T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\left|\left[\psi(t_2)-\psi(a)\right]^{1-\varsigma_1}(\mathcal{F}_1 x)(t_2) - \left[\psi(t_1)-\psi(a)\right]^{1-\varsigma_1}(\mathcal{F}_1 x)(t_1)\right| \\
&\quad\leq \frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t_2,a)}\frac{1}{\Gamma(\theta_1)}\int_{t_1}^{t_2}\mathcal{H}_{\psi}^{\theta_1}(t_2,s)\left|f_1\left(s,x(s)\right)\right|ds \\
&\qquad+\frac{1}{\Gamma(\theta_1)}\left|\int_a^{t_1}\frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t_2,a)}\mathcal{H}_{\psi}^{\theta_1}(t_2,s) - \frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t_1,a)}\mathcal{H}_{\psi}^{\theta_1}(t_1,s)\right|\left|f_1\left(s,x(s)\right)\right|ds \\
&\quad\leq \frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t_2,a)}\left(I_{t_1}^{\theta_1,\psi}\left[\psi(s)-\psi(a)\right]^{\varsigma_1-1}\right)(t_2)\left\|f_1\left(\cdot,x(\cdot)\right)\right\|_E \\
&\qquad+\left|\frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t_2,a)}\left(I_a^{\theta_1,\psi}\left[\psi(s)-\psi(a)\right]^{\varsigma_1-1}\right)(t_2)\right. \\
&\qquad\left.-\frac{1}{\mathcal{K}_{\psi}^{\varsigma_1}(t_1,a)}\left(I_a^{\theta_1,\psi}\left[\psi(s)-\psi(a)\right]^{\varsigma_1-1}\right)(t_1)\right|\left\|f_1\left(\cdot,x(\cdot)\right)\right\|_E
\end{aligned}$$

which leads to

$$\left\|(\mathcal{F}_1x)(t_2)-(\mathcal{F}_1x)(t_1)\right\|_E \to 0, \quad \text{as } t_2 \to t_1.$$

Again applying the same reasoning, we have

$$\left\|\mathcal{F}_2y(t_2)-\mathcal{F}_2y(t_1)\right\|_E \to 0, \quad \text{as } t_2 \to t_1.$$

This exhibits that $\mathcal{F}(\mathcal{K})$ is equicontinuous. Stages 1–3 show that $\mathcal{F}$ is relatively compact on $\mathcal{K}$. By $E(=C_{1-\varsigma;\psi})$ type Arzelá–Ascoli theorem, $\mathcal{F}$ is compact on $\mathcal{K}$.
Stage(4): $\mathcal{Q}$ is a contraction operator.
Let $(y,x),(y^*,x^*) \in E \times E$ and $t \in (a,T]$. Then, by applying (H_2), we easily get

$$\left\|(\mathcal{G}_1x)-(\mathcal{G}_1x^*)\right\|_E \le \Lambda_{f_1}\left\|x-x^*\right\|_E$$

and

$$\left\|(\mathcal{G}_2y)-(\mathcal{G}_2y^*)\right\|_E \le \Lambda_{f_2}\left\|y-y^*\right\|_E.$$

Since $\Lambda_{f_1},\Lambda_{f_2} < 1$, $\mathcal{G}$ is a contraction mapping. Using Theorem 1, we see that $\mathcal{N}$ has at least one fixed point, which is the corresponding solution of ψ-Hilfer coupled system (3)–(4). □

$$\left|D_{a^+}^{\theta_1,\eta_1,\psi}\tilde{y}(t) - f_1\left(t, \tilde{x}(t)\right)\right| \le \epsilon_1,$$

(27)

$$\left|D_{a^+}^{\theta_2,\eta_2,\psi}\tilde{x}(t) - f_2\left(t, \tilde{y}(t)\right)\right| \le \epsilon_2,$$

(28)

then there exists $(y, x) \in E \times E$ satisfying coupled system (3) with the following coupled boundary conditions:

$$\begin{cases} y(T) = \tilde{y}(T), \\ x(T) = \tilde{x}(T), \end{cases}$$

(29)

complying with

$$\left\|(\tilde{y}, \tilde{x}) - (y, x)\right\|_{E\times E} \le \lambda\epsilon.$$

Definition 6

The ψ-Hilfer coupled system (3)–(4) is G-U-H stable if there exists $\varphi = (\varphi_{f_1}, \varphi_{f_2}) \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\varphi(0) = (\varphi_{f_1}(0), \varphi_{f_2}(0)) = (0, 0)$ such that for some $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and for each solution $(\tilde{y}, \tilde{x}) \in E \times E$ of

(i) $|h_1(t)| \leq \epsilon_1$ and $|h_2(t)| \leq \epsilon_2$ for $t \in (a,T]$,
(ii) For $t \in (a,T]$,

$$\begin{cases} D_{a^+}^{\theta_1,\eta_1,\psi}\tilde{y}(t) = f_1(t,\tilde{x}(t)) + h_1(t), \\ D_{a^+}^{\theta_2,\eta_2,\psi}\tilde{x}(t) = f_2(t,\tilde{y}(t)) + h_2(t). \end{cases}$$

Lemma 8

Let $(\tilde{y},\tilde{x}) \in E \times E$ *be the solution of inequalities* (27)–(28). *Then* $(y,x) \in E \times E$ *is the solution of the following fractional integral inequalities*:

$$\left|\tilde{y}(t) - Z_{\tilde{x}} - \frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_\psi^{\theta_1}(t,s) f_1\left(s,\tilde{x}(s)\right) ds\right| \leq \frac{2\epsilon_1 \mathcal{K}_\psi^{\theta_1+1}(T,a)}{\Gamma(\theta_1+1)}$$

and

$$\left|\tilde{x}(t) - Z_{\tilde{u}} - \frac{1}{\Gamma(\theta_2)}\int_a^t \mathcal{H}_\psi^{\theta_2}(t,s) f_2\left(s,\tilde{y}(s)\right) ds\right| \leq \frac{2\epsilon_2 \mathcal{K}_\psi^{\theta_2+1}(T,a)}{\Gamma(\theta_2+1)},$$

where

$$Z_{\tilde{x}} := \frac{\mathcal{K}_\psi^{\varsigma_1}(t,a)}{\mathcal{K}_\psi^{\varsigma_1}(T,a)}\left[w_1 - \frac{1}{\Gamma(\theta_1)}\int_a^T \mathcal{H}_\psi^{\theta_1}(T,s) f_1\left(s,\tilde{x}(s)\right) ds\right]$$

and

Thanks to Theorem 15, the solution of (30) is defined by

$$\tilde{y}(t)=\begin{cases}\frac{\mathcal{K}_\psi^{\varsigma_1}(t,a)}{\mathcal{K}_\psi^{\varsigma_1}(T,a)}[w_1-\frac{1}{\Gamma(\theta_1)}\int_a^T\mathcal{H}_\psi^{\theta_1}(T,s)f_1(s,\tilde{x}(s))\,ds]\\ \quad+\frac{\mathcal{K}_\psi^{\varsigma_1}(t,a)}{\mathcal{K}_\psi^{\varsigma_1}(T,a)}[w_1-\frac{1}{\Gamma(\theta_1)}\int_a^T\mathcal{H}_\psi^{\theta_1}(T,s)h_1(s)\,ds]\\ \quad+\frac{1}{\Gamma(\theta_1)}\int_a^t\mathcal{H}_\psi^{\theta_1}(t,s)h_1(s)\,ds\\ \quad+\frac{1}{\Gamma(\theta_1)}\int_a^t\mathcal{H}_\psi^{\theta_1}(t,s)f_1(s,\tilde{x}(s))\,ds\end{cases}$$

(31)

and

$$\tilde{x}(t)=\begin{cases}\frac{\mathcal{K}_\psi^{\varsigma_2}(t,a)}{\mathcal{K}_\psi^{\varsigma_2}(T,a)}[w_2-\frac{1}{\Gamma(\theta_2)}\int_a^T\mathcal{H}_\psi^{\theta_2}(T,s)f_2(s,\tilde{y}(s))\,ds]\\ \quad+\frac{\mathcal{K}_\psi^{\varsigma_2}(t,a)}{\mathcal{K}_\psi^{\varsigma_2}(T,a)}[w_2-\frac{1}{\Gamma(\theta_1)}\int_a^T\mathcal{H}_\psi^{\theta_2}(T,s)h_1(s)\,ds]\\ \quad+\frac{1}{\Gamma(\theta_2)}\int_a^t\mathcal{H}_\psi^{\theta_2}(t,s)h_1(s)\,ds\\ \quad+\frac{1}{\Gamma(\theta_2)}\int_a^t\mathcal{H}_\psi^{\theta_2}(t,s)f_2(s,\tilde{y}(s))\,ds.\end{cases}$$

(32)

It follows from (31) and (32) with using Lemma 8 that

$$\left|\tilde{y}(t)-Z_{\tilde{x}}-\frac{1}{\Gamma(\theta_1)}\int_a^t\mathcal{H}_\psi^{\theta_1}(t,s)f_1(s,\tilde{x}(s))\,ds\right|$$

□

Theorem 6

Under the assumptions of Theorem 4*, if* $1-\mathcal{C}_{\theta_1}\mathcal{C}_{\theta_2}\neq 0$, *then the* ψ*-Hilfer coupled system* (3)–(4) *will be U-H and G-U-H stable in* $E\times E$*, where* $\mathcal{C}_{\theta_1}:=\frac{\Gamma(\varsigma_1)L_1}{\Gamma(\varsigma_1+\theta_1)}\mathcal{K}_{\psi}^{\theta_1+1}(T,a)$ *and* $\mathcal{C}_{\theta_2}:=\frac{\Gamma(\varsigma_2)L_2}{\Gamma(\varsigma_2+\theta_2)}\mathcal{K}_{\psi}^{\theta_2+1}(T,a)$.

Proof

Let $(\tilde{y},\tilde{x})\in E\times E$ be the solution of coupled system (30) and $(y,x)\in E\times E$ be a unique solution of the ψ-Hilfer coupled system (3)–(4) with the conditions

$$\begin{cases} y(T)=\tilde{y}(T)=w_1, \\ x(T)=\tilde{x}(T)=w_2. \end{cases}$$

(33)

That is,

$$y(t)=Z_x+\frac{1}{\Gamma(\theta_1)}\int_a^t \mathcal{H}_{\psi}^{\theta_1}(t,s)f_1\big(s,x(s)\big)\,ds$$

(34)

and

and

$$x(t) = Z_{\tilde{y}} + \frac{1}{\Gamma(\theta_2)} \int_a^t \mathcal{H}_\psi^{\theta_2}(t,s) f_2\big(s, y(s)\big)\,ds.$$

Through Lemma 8, we arrive at

$$\left|\tilde{y}(t) - Z_{\tilde{x}} - \frac{1}{\Gamma(\theta_1)} \int_a^t \mathcal{H}_\psi^{\theta_1}(t,s) f_1\big(s, \tilde{x}(s)\big)\,ds\right| \le \mathcal{A}_1 \epsilon_1$$

(36)

and

$$\left|\tilde{x}(t) - Z_{\tilde{u}} - \frac{1}{\Gamma(\theta_2)} \int_a^t \mathcal{H}_\psi^{\theta_2}(t,s) f_2\big(s, \tilde{y}(s)\big)\,ds\right| \le \mathcal{A}_2 \epsilon_2,$$

(37)

where $\mathcal{A}_1 = \frac{2\mathcal{K}_\psi^{\theta_1+1}(T,a)}{\Gamma(\theta_1+1)}$ and $\mathcal{A}_2 = \frac{2\mathcal{K}_\psi^{\theta_2+1}(T,a)}{\Gamma(\theta_2+1)}$.
Thus, by (H_2) and inequalities (36), (37), we reach

$$\begin{aligned}
&\big|\tilde{y}(t) - y(t)\big| \\
&\quad = \left|\tilde{y}(t) - Z_{\tilde{x}} - \frac{1}{\Gamma(\theta_1)} \int_a^t \mathcal{H}_\psi^{\theta_1}(t,s) f_1\big(s, \tilde{x}(s)\big)\,ds\right| \\
&\qquad + \frac{1}{\Gamma(\theta_1)} \int_a^t \mathcal{H}_\psi^{\theta_1}(t,s) \big|f_1\big(s, \tilde{x}(s)\big) - f_1\big(s, x(s)\big)\big|\,ds
\end{aligned}$$

where $\overline{\mathcal{A}_1} := \frac{\mathcal{A}_1}{\mathcal{K}_\psi^{\varsigma_1}(T,a)}$.

Similarly, we have

$$\|\tilde{x} - x\|_E \leq \overline{\mathcal{A}_2}\epsilon_2 + \mathcal{C}_{\theta_2}\|\tilde{y} - y\|_E, \tag{39}$$

where $\overline{\mathcal{A}_2} := \frac{\mathcal{A}_2}{\mathcal{K}_\psi^{\varsigma_2}(T,a)}$. Inequalities (38) and (39) can be rewritten again as follows:

$$\begin{cases} \|\tilde{y} - y\|_E - \mathcal{C}_{\theta_1}\|\tilde{x} - x\|_E \leq \overline{\mathcal{A}_1}\epsilon_1, \\ \|\tilde{x} - x\|_E - \mathcal{C}_{\theta_2}\|\tilde{y} - y\|_E \leq \overline{\mathcal{A}_2}\epsilon_2. \end{cases} \tag{40}$$

Now, we will represent the relations in (40) as matrices as follows:

$$\begin{pmatrix} 1 & -\mathcal{C}_{\theta_1} \\ -\mathcal{C}_{\theta_2} & 1 \end{pmatrix} \begin{pmatrix} \|\tilde{y} - y\|_E \\ \|\tilde{x} - x\|_E \end{pmatrix} \leq \begin{pmatrix} \overline{\mathcal{A}_1}\epsilon_1 \\ \overline{\mathcal{A}_2}\epsilon_2 \end{pmatrix}.$$

After simple computations of the above inequality, we can write

$$\begin{pmatrix} \|\tilde{y} - y\|_E \end{pmatrix} \leq \begin{pmatrix} \frac{1}{\Delta} & \frac{\mathcal{C}_{\theta_1}}{\Delta} \end{pmatrix} \times \begin{pmatrix} \overline{\mathcal{A}_1}\epsilon_1 \end{pmatrix}$$

$$\|\tilde{y}-y\|_E+\|\tilde{x}-x\|_E \le \left(\frac{\overline{\mathcal{A}_1}+\mathcal{C}_{\theta_2}\overline{\mathcal{A}_1}}{\Delta}\right)\epsilon_1+\left(\frac{\overline{\mathcal{A}_2}+\mathcal{C}_{\theta_1}\overline{\mathcal{A}_2}}{\Delta}\right)\epsilon_2.$$

For $\epsilon=\max\{\epsilon_1,\epsilon_2\}$ and

$$\lambda=\left(\frac{\overline{\mathcal{A}_1}+\mathcal{C}_{\theta_2}\overline{\mathcal{A}_1}+\overline{\mathcal{A}_2}+\mathcal{C}_{\theta_1}\overline{\mathcal{A}_2}}{\Delta}\right),$$

we obtain

$$\left\|(\tilde{y},\tilde{x})-(y,x)\right\|_{E\times E}\le\lambda\epsilon. \tag{41}$$

This proves that the ψ-Hilfer coupled system (3)–(4) is U-H stable.
Moreover, we could put into writing inequality (41) as

$$\left\|(\tilde{y},\tilde{x})-(y,x)\right\|_{E\times E}\le\varphi(\epsilon),$$

where $\varphi(\epsilon)=\lambda\epsilon$ with $\varphi(0)=0$. This shows that the ψ-Hilfer coupled system (3)–(4) is G-U-H stable. □

5 Examples

(42)

Set

$$f_1(t,u)=\frac{1}{10e^{t+1}}\frac{u}{1+u}+\frac{1}{\sqrt{t+1}},\quad u,v\in\mathbb{R},t\in(0,1]$$

and

$$f_2(t,v)=\frac{1}{(t+6)^2}\frac{v}{1+v}+\frac{1}{2\sqrt{t+1}}\quad u,v\in\mathbb{R},t\in(0,1].$$

Then

$$C^{\eta_1(1-\theta_1)}_{1-\varsigma_1;\psi}[0,1]=C^0_{\frac{1}{2};e^{\frac{t}{3}}}[0,1]=\{f_1:(0,1]\times\mathbb{R}\to\mathbb{R};(e^{\frac{t}{3}}-1)^{\frac{1}{2}}f_1\in C[0,1]\},$$
$$C^{\eta_2(1-\theta_2)}_{1-\varsigma_2;\psi}[0,1]=C^0_{\frac{1}{2};e^{\frac{t}{3}}}[0,1]=\{f_2:(0,1]\times\mathbb{R}\to\mathbb{R};(e^{\frac{t}{3}}-1)^{\frac{1}{2}}f_2\in C[0,1]\},$$

with $\theta_1=\theta_2=\frac{1}{2},\eta_1=\eta_2=0,\varsigma_1=\varsigma_2=\frac{1}{2},\psi(t)=e^{\frac{t}{3}}$, and $(a,T]=(0,1]$. Evidently, the functions $f_1,f_2\in C_{\frac{1}{2};e^{\frac{t}{3}}}[0,1]$. So, hypothesis (H_1) holds. For $t\in(0,1]$ and $u,u^*,v,v^*\in\mathbb{R}$, we have

$$|f_1(t,u)-f_1(t,u^*)|\le\frac{1}{10e}|u-u^*|,$$

, where $\mathcal{C}_{\theta_1} = \frac{\sqrt{\pi(\sqrt[3]{e}-1)}}{10e}$ and $\mathcal{C}_{\theta_2} = \frac{\sqrt{\pi(\sqrt[3]{e}-1)}}{36}$. Therefore, from Theorem 6, coupled system (42) is U-H and G-U-H stable.

Example 2

A particular case, for $\theta_1 = \theta_2 = \frac{1}{2}, \eta_1 = \eta_2 = 1$, and $\psi(t) = t$, the coupled system for ψ-Hilfer FDE (42) reduces to the following coupled system for Caputo FDE:

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2}} y(t) = \frac{1}{10e^{t+1}} \frac{|x(t)|}{1+|x(t)|} + \frac{1}{\sqrt{t+1}}, & t \in [0,1], \\ {}^C D_{0^+}^{\frac{1}{2}} x(t) = \frac{1}{(t+2)^2} \frac{|y(t)|}{1+|y(t)|} + \frac{1}{2\sqrt{t+1}}, & t \in [0,1], \\ y(1) = 1, \qquad x(1) = 2. \end{cases}$$

(43)

Clearly, the function $f_1, f_2 \in C[0,1]$. Hence (H_1) holds. Also, hypothesis (H_2) is satisfied with $L_1 = \frac{1}{10e}$ and $L_2 = \frac{1}{36}$. Via some straightforward computations, we see that $A_1 = 1$, $A_2 = \frac{1}{2}$, $\Lambda_{f_1} = \frac{2}{5e\sqrt{\pi}} < 1$, $\Lambda_{f_2} = \frac{1}{9\sqrt{\pi}} < 1$, $\Delta_{f_1} = 1 + \frac{4}{\sqrt{\pi}}$, and $\Delta_{f_2} = 2 + \frac{2}{\sqrt{\pi}}$. Thus all the suppositions in Theorem 4 are satisfied. An application of Theorem 4 shows that problem (43) has a unique solution in $C[0,1]$. Moreover, since $\Delta = 1 - \mathcal{C}_{\theta_1}\mathcal{C}_{\theta_1} = 1 - \frac{1}{90e\pi} \neq 0$, where $\mathcal{C}_{\theta_1} = \frac{1}{5e\sqrt{\pi}}$ and

(44)

Through comparing coupled system (3)–(4) with (44), we have $\theta_1 = \theta_2 = \frac{1}{4}, \eta_1 = \eta_2 = \frac{1}{3}, \varsigma_1 = \varsigma_2 = \frac{1}{2}$, $\psi(t) = \log t, a = 1, T = e, w_1 = 1, w_2 = 2$ and $f_1(t,u) = \frac{1}{5+2^t}\frac{u^2+u}{1+u} + \sqrt{2}$ and $f_2(t,v) = \frac{e^{-\log t}}{e^{\log t}+8}\frac{v}{1+v} + \frac{3}{2}$ for $u, v \in \mathbb{R}, t \in (1,e]$. Evidently, the functions $f_1, f_2 \in C_{\frac{1}{2};\log}[1,e]$. So, condition (H_1) holds. It is simple to verify that

$$|f_1(t,u) - f_1(t,u^*)| \le \frac{1}{7}|u - u^*|, \quad t \in (1,e], u, u^* \in \mathbb{R},$$
$$|f_2(t,v) - f_2(t,v^*)| \le \frac{1}{9}|v - v^*|, \quad t \in (1,e], v, v^* \in \mathbb{R}.$$

Thus, hypothesis (H_2) is satisfied with $L_1 = \frac{1}{7}$ and $L_2 = \frac{1}{9}$. Through some easy calculations, we infer that $A_1 = \sqrt{2}, A_2 = \frac{3}{2}, \Lambda_{f_1} \approx 0.4 < 1, \Lambda_{f_2} \approx 0.3 < 1$, $\Delta_{f_1} \approx 4.1$, and $\Delta_{f_2} \approx 5.3$. Since every supposition in Theorem 4 is satisfied, problem (44) has a unique solution in $C^{\frac{1}{2}}_{\frac{1}{2},\log}[1,e]$. Further, as shown in Theorem 6, for every $\epsilon = \max(\epsilon_1, \epsilon_2) > 0$, if $(\tilde{y}, \tilde{x}) \in E \times E$ satisfies

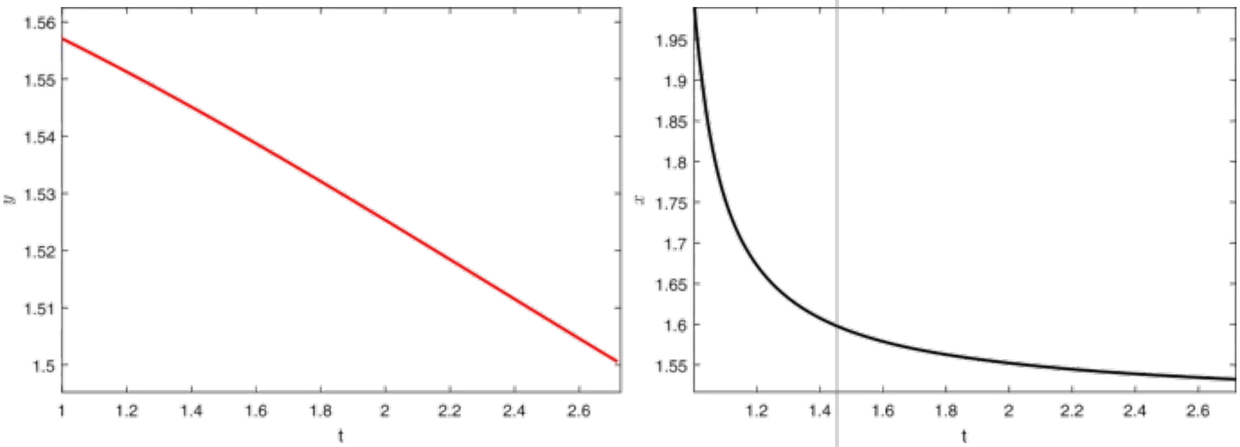
$$\begin{cases} |D^{\theta_1,\eta_1,\log}_{a^+}\tilde{y}(t) - \frac{1}{5+2^t}\frac{\tilde{x}^2(t)+|\tilde{x}(t)|}{1+|\tilde{x}(t)|} + \sqrt{2}| \le \epsilon_1, \\ |D^{\theta_2,\eta_2,\log}_{a^+}\tilde{x}(t) - \frac{e^{-\log t}}{\dots}\frac{|\tilde{y}(t)|}{\dots} + \frac{3}{\dots}| \le \dots \end{cases}$$

$\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2} = \frac{2}{\Gamma(\frac{5}{4})}, \mathcal{C}_{\theta_1} = \frac{\sqrt{\pi}}{7\Gamma(\frac{3}{4})}, \mathcal{C}_{\theta_2} = \frac{\sqrt{\pi}}{9\Gamma(\frac{3}{4})}$, and

$\Delta = 1 - \mathcal{C}_{\theta_1}\mathcal{C}_{\theta_1} = 1 - \frac{\pi}{63\Gamma(\frac{3}{4})^2} \neq 0.$

Hence coupled system (44) is U-H and G-U-H stable. Some graphical presentation of Example 3 is shown in Fig. 1.

Figure 1



Graphical representation of approximate solution for Example 3 at various fractional order of coupled system (44)

Abbreviations

FDEs:

Fractional differential equations

TVPs:

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None of the authors has conflict of interests regarding this work.

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