Existence and Ulam–Hyers–Mittag-Leffler stability results of -Hilfer nonlocal Cauchy problem

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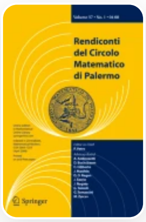

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Abstract

The main aim of this article is to investigate some new sufficient conditions of the existence and uniqueness results of ψ-Hilfer fractional functional differential equations with nonlocal condition and finite delay. Further, Ulam–Hyers–Mittag-Leffler stability of solutions to a proposed problem is discussed. Krasnoselskii fixed point theorem, Picard operator method and Gronwall's inequality lemma are the main tools of our analysis. An illustrative example is provided in support of the results obtained.

1 Introduction

Fractional differential equations have recently confirmed to be significant tools in modeling many phenomena in various fields of engineering and science, since their nonlocal property is suitable to describe memory phenomena such as nonlocal elasticity, polymers, propagation in complex medium, biological, electrochemistry, porous media, viscoelasticity, electromagnetics, see [4, 5, 10] and references therein. There has been considerable growth in ordinary and partial differential equations involving Riemann–Liouville, Caputo, and Hilfer fractional derivatives in recent years. For details, we refer the reader to monographs of Kilbas et al. [12], Samko et al. [20], Hilfer [8], and Podlubny [18].

The stability problem related to functional differential equations was originally emerged by Ulam [29, 30]. The Ulam's problem is given as "under what conditions there exist an additive mapping near an approximately additive mapping". Hyers [9] answered on Ulam's question about the additive mapping in Banach spaces and became this type of stability called Ulam–Hyers stability. The concept of Ulam–Hyers Stability was extended via inserting new function variables provided by Rassias [19]. Ulam-stability, Ulam–

Hyers stability, and Ulam−Hyers−Rassias stability, these labels have become famous today in literature.

Many numbers of research work can be found in the literature that deals with the different types of stabilities such as Ulam−Hyers, generalized Ulam−Hyers, Ulam−Hyers−Rassias and generalized Ulam−Hyers−Rassias of different kinds of classical differential and fractional differential equations. For some results and recent development on these types of stabilities, we refer the reader to a series of papers [1, 7, 13, 21, 32] and the references given therein.

In particular, the Ulam-type stability of delay differential equations was investigated by Otrocol et al. [17] and Kucche et al. [13]. In the same context, the Ulam−Hyers−Mittag-Leffler stability of delay fractional differential equations was investigated by Wang et al. [31], and Liu et al. [15].

Recently, Sousa and Oliveira [22] proposed a ψ-Hilfer fractional operator and apply it in improve and extend some previous works dealing with the Hilfer fractional derivative operator, exclusively [7, 8]. The authors in [24], proposed a generalized Gronwall inequality through the fractional integral with respect to another function in the concept of Hilfer. Also, in [27] they presented a Leibniz type rule for the ψ-Hilfer fractional derivative operator in two forms.

On the other hand, Teodoro et al. [28] introduced a systematic classification for fractional calculus due to an increasing number of proposals and definitions of operators in this scope. He considered that many of the definitions that emerged in the literature can not be considered as fractional derivatives. Moreover, the author analyzed a list of expressions to have a general overview of the concept of fractional integrals and derivatives.

For the existence and uniqueness results of different classes of initial value problem for fractional differential equations involving ψ-Hilfer derivative operator, one can see [2, 3, 14, 16, 23, 25, 26], and the references given therein.

Motivated by the above-mentioned works, in the present paper, using Picard operator method and Gronwall's inequality, we investigate the existence, uniqueness and Ulam−

Hyers–Mittag-Leffler stability results for ψ-Hilfer fractional functional differential equation

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\beta;\psi} x(t) = f(t, x(t), x(\mathfrak{g}(t))), t \in (0, b] \\ I_{0^+}^{1-\gamma;\psi} x(0^+) = \sum_{i=1}^{k} c_i x(\tau_i), \qquad \tau_i \in (0, b) \\ x(t) = \varphi(t), \qquad t \in [-r, 0] \end{cases}$$

(1.1)

where ${}^{H}D_{0^+}^{\alpha,\beta;\psi}(\cdot)$ is ψ-Hilfer fractional derivative operator of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$, $I_{0^+}^{1-\gamma,\psi}(\cdot)$ is ψ-fractional integral in Riemann–Liouville sense of order $1-\gamma$ $(\gamma = \alpha + \beta(1-\alpha), 0 < \gamma < 1, \tau_i, i = 1, 2, \ldots, k$ are prefixed points satisfying $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_i < b,$ and $c_i \in \mathbb{R}, \varphi \in C[-r, 0],$ $f : (0, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $\mathfrak{g} \in C(0, b] \to [-r, b]$ with $\mathfrak{g}(t) \leq t, r > 0$.

The main contributions are as follows: In Sect. 2, we recall some notations, definitions, and preliminary facts which are used throughout this paper. In Sect. 3, we obtain the integral equation which is equivalent to the problem (1.1). Moreover, we provide sufficient conditions to guarantee the existence and uniqueness of solutions for the problem (1.1) by means of Banach fixed point theorem, Krasnoselskii fixed point theorem and Picard operator method. Using Gronwall inequality, we prove the Ulam–Hyers–Mittag-Leffler stability of the problem (1.1) in Sect. 4. An illustrative example is given in Sect. 5, and the conclusion is presented in the last Section.

2 Preliminaries

In this section, we recall some basic definitions of the fractional calculus theory which are used throughout this paper. Let $[a, b] \subset \mathbb{R}^+$ with $(0 < a < b < \infty)$, and let $C[a, b]$ be the Banach space of continuous function $\theta : [a, b] \to \mathbb{R}$ with the norm $\|\theta\|_{C[a,b]} = \max\{|\theta(t)| : a \leq t \leq b\}$. The weighted space $C_{1-\gamma;\psi}[a, b]$ of continuous function $\theta : [a, b] \to \mathbb{R}$ is defined by (see [22])

$$C_{1-\gamma;\psi}[a, b] = \left\{\theta : (a, b] \to \mathbb{R}; [\psi(t) - \psi(a)]^{1-\gamma} \theta(t) \in C[a, b]\right\}.$$

Obviously, $C_{1-\gamma;\psi}\ [a,b]$ is a Banach space endowed with the norm

$$\|\theta\|_{C_{1-\gamma;\psi[a,b]}} = \max_{t\in[a,b]} \left|[\psi(t)-\psi(a)]^{1-\gamma}\theta(t)\right|.$$

Definition 2.1

(see [11]) Let $\alpha > 0, \beta > 0$. Then the two-parameters Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{R}.$$

(2.1)

If $\beta = 1$, the one-parameter Mittag-Leffler function defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z \in \mathbb{R}, \ \alpha > 0.$$

(2.2)

Definition 2.2

(see [22]) Let $\alpha > 0, \theta \in L_1\ [a,b]$ and $\psi \in C^1\ [a,b]$ be an increasing function with $\psi'(t) \neq 0$, for all $t \in [a,b]$. Then, the left-sided ψ-Riemann–Liouville fractional integral of a function θ is defined by

$$I_{a^+}^{\alpha,\psi}\theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\theta(s)ds.$$

Definition 2.3

(see [22]) Let $n-1 < \alpha < n, (n = [\alpha]+1)$, and $\theta, \psi \in C^n\ [a,b]$ be two functions

Loading [MathJax]/jax/output/SVG/fonts/TeX/Main/Italic/Main.js the left-sided ψ-Riemann–

Liouville fractional (ψ-Caputo) derivative of a function θ of order α is defined by

$$D_{a^+}^{\alpha,\psi}\theta(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\psi}\theta(t),$$

and

$$^C D_{a^+}^{\alpha,\psi}\theta(t) = I_{a^+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \theta(t),$$

respectively.

Definition 2.4

(see [22]) Let $n-1<\alpha<n\in\mathbb{N}$, and $\theta,\psi\in C^n[a,b]$ be two functions such that ψ is increasing and $\psi'(t)\neq 0$, for all $t\in[a,b]$. The left-sided ψ-Hilfer fractional derivative of a function θ of order α and type $0\leq\beta\leq 1$ is defined by

$$\begin{aligned} ^H D_{a^+}^{\alpha,\beta,\psi}\theta(t) &= I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}\theta(t) \\ &= I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi}\theta(t), (\gamma=\alpha+n\beta-\alpha\beta). \end{aligned}$$

In particular, if $0<\alpha<1, 0\leq\beta\leq 1$ and $\gamma=\alpha+\beta-\alpha\beta$. Then

$$\begin{aligned} ^H D_{a^+}^{\alpha,\beta,\psi}\theta(t) &= I_{a^+}^{\beta(1-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) I_{a^+}^{(1-\beta)(1-\alpha);\psi}\theta(t) \\ &= I_{a^+}^{\beta(1-\alpha);\psi} D_{a^+}^{\gamma;\psi}\theta(t). \end{aligned}$$

(2.3)

Now, we introduce the following spaces

$$C^{\alpha,\beta}_{1-\gamma;\psi}[a,b] = \{x \in C_{1-\gamma;\psi}[a,b], D^{\alpha,\beta;\psi}_{a^+}x \in C_{1-\gamma;\psi}[a,b]\},$$

and

$$C^{\gamma}_{1-\gamma;\psi}[a,b] = \{x \in C_{1-\gamma;\psi}[a,b], D^{\gamma;\psi}_{a^+}x \in C_{1-\gamma;\psi}[a,b]\},$$

(2.4)

we note that $C^{\gamma}_{1-\gamma;\psi}[a,b] \subset C^{\alpha,\beta}_{1-\gamma;\psi}[a,b]$.

Lemma 2.1

(see [22]) *Let* $\alpha > 0, \beta > 0, \gamma = \alpha + \beta - \alpha\beta$ *and* $\theta \in C^{\gamma}_{1-\gamma;\psi}[J,\mathbb{R}]$. *Then*

$$I^{\gamma;\psi}_{a^+} D^{\gamma;\psi}_{a^+}\theta = I^{\alpha;\psi}_{a^+}\, {}^{H}D^{\alpha,\beta;\psi}_{a^+}\theta,$$

and

$$D^{\gamma;\psi}_{a^+} I^{\alpha;\psi}_{a^+}\theta = D^{\beta(1-\alpha);\psi}_{a^+}\theta.$$

Theorem 2.1

(see [22]) *Let* $\theta \in C^1[a,b], 0 < \alpha < 1$ *and* $0 \le \beta \le 1$. *Then*

$${}^{H}D^{\alpha,\beta,\psi}_{a^+} I^{\alpha,\psi}_{a^+}\theta(t) = \theta(t).$$

Theorem 2.2

(see [12]) *Let* $\alpha, \gamma > 0$. *Then*

$$[illegible] \Gamma(\gamma) [illegible] \quad t > a.$$

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Theorem 2.3

(see [22]) *Let* $0<\alpha<1, 0\leq\beta\leq1, \theta\in C_{1-\gamma,\psi}[a,b]$ *and* $I_{a^+}^{1-\gamma;\psi}\theta\in C^1_{1-\gamma,\psi}[a,b]$. *Then we have*

$$I_{a^+}^{\gamma;\psi}D_{a^+}^{\gamma,\psi}\theta(t)=\theta(t)-\frac{I_{a^+}^{1-\gamma;\psi}\theta(a)}{\Gamma(\gamma)}(\psi(t)-\psi(a))^{\gamma-1}.$$

Theorem 2.4

(see [22]) *Let* $0\leq\gamma<\alpha$ *and* $\theta\in C_{1-\gamma,\psi}[a,b]$. *Then*

$$I_{a^+}^{\alpha;\psi}\ \theta(a)=\lim_{t\to a^+}I_{a^+}^{\alpha;\psi}\theta(t)=0.$$

Theorem 2.5

(see [6]) (Krasnoselskii fixed point theorem) *Let K be closed, convex, bounded and nonempty subset of a Banach space X and* Φ^1, Φ^2 *be two operators such that*

1. $\Phi^1u+\Phi^2v\in X$ for all $u,v\in X$.
2. Φ^1 is compact and continuous.
3. Φ^2 is contraction mapping.

Then there exists $z\in K$ *such that* $z=\Phi^1z+\Phi^2z$.

Definition 2.5

(see [31]) Let (X, d) be a metric space. Now $T:X\to X$ is a Picard operator if there exists $u^*\in X$ such that $F_T=u^*$ where $F_T=\{u\in X: T(u)=u\}$ is the fixed point set of T, and the sequence $(T^n(u_0))_{n\in\mathbb{N}}$ converges to u^* for all $u_0\in X$.

Lemma 2.2

(see [31]) *Let* $(X,d,\leq)$ *be an ordered metric space, and let* $T:X\to X$ *be an increasing*

Loading [MathJax]/jax/output/SVG/fonts/TeX/Main/Italic/Main.js *mplies* $u\leq u_T^*$.

Lemma 2.3

(see [24]) (Generalized Gronwall's Inequality Lemma) *Let* $\alpha > 0$ *and* x, y *be two nonnegative function locally integrable on* $[a, b]$. *Assume that* g *is nonnegative and nondecreasing, and let* $\psi \in C^1[a, b]$ *an increasing function such that* $\psi^{'}(t) \neq 0$ *for all* $t\in [a,b]$. *If*

$$\begin{aligned} x(t)\le y(t)+g(t)\int _{a}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}x(s)ds,\ \ t\in [a,b]. \end{aligned}$$

Then

$$\begin{aligned} x(t)\le y(t)+\int _{a}^{t}\sum \limits _{n=1}^{\infty }\frac{\left[g(t)\Gamma (\alpha)\right] ^{n}}{\Gamma (n\alpha)}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{n\alpha -1}y(s)ds,\ \ t\in [a,b]. \end{aligned}$$

If y be a nondecreasing function on $[a, b]$. Then

$$\begin{aligned} x(t)\le y(t)E_{\alpha }\left\{ g(t)\Gamma (\alpha)\left[\psi (t)-\psi (a) \right] ^{\alpha }\right\} ,\ \ t\in [a,b]. \end{aligned}$$

Lemma 2.4

(see [24]) *Let* $f:(0,b]\times {\mathbb {R}} \rightarrow {\mathbb {R}}$ *be a continuous function. Then the following problem*

$$\begin{aligned} \begin{array}{l} ^{H}D_{0^{+}}^{\alpha ,\beta ;\psi }x(t)=f(t,x(t)),t\in (0,b] \\ I_{0^{+}}^{1-\gamma ;\psi }x(0)=x_{0},\qquad \qquad \qquad \ \end{array} \end{aligned}$$

is equivalent to integral equation

$$\begin{aligned} x(t)=\frac{(\psi (t)-\psi (0))^{\gamma -1}}{\Gamma (\gamma)}x_{0}+\frac{1}{ \Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,x(s))ds. \end{aligned}$$

(2.5)

3 Main results

In this section, we prove the existence and uniqueness of solution for the problem (1.1). For our analysis the following hypotheses should be satisfied. $(\hbox {H}_{1})$ $f:(0,b]\times {\mathbb {R}} \times {\mathbb {R}} \rightarrow {\mathbb {R}}$ be a function such that $f(\cdot ,x(\cdot),x({\mathfrak {g}}(\cdot))\in C_{1-\gamma ;\psi }^{\beta (1-\alpha)}[0,b]$ for any $x\in C[-r,b]\cap C_{1-\gamma ;\psi }[0,b]$ and ${\mathfrak {g}}:(0,b]\rightarrow [-r,b]$ with ${\mathfrak {g}}(t)\le t,\ r>0.$

$(\hbox {H}_{2})$ There exists ${\mathscr {M}}_{f}\in \left(0,\frac{1}{2}\right)$ such that

$$\begin{aligned} \left| f(t,u_{1},u_{2})-f(t,v_{1},v_{2})\right| \le {\mathscr {M}}_{f} \left[\left| u_{1}-v_{1}\right| +\left| u_{2}-v_{2}\right| \right] , \end{aligned}$$

for all $\ t\in (0,b], u_{i},v_{i}\in {\mathbb {R}} ,i=1,2.$

$(\hbox {H}_{3})$ The following inequality holds

$$\begin{aligned} \Omega :=\frac{2{\mathscr {M}}_{f}{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha) }\left[\frac{1}{\Gamma (\gamma)(1-\mu)}\sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha }+\left(\psi (b)-\psi (0\right) ^{\alpha }\right] <1, \end{aligned}$$

where

$$\begin{aligned} \mu :=\sum \limits _{i=1}^{k}c_{i}\frac{(\psi (\tau _{i})-\psi (0))^{\gamma -1} }{\Gamma (\gamma)}. \end{aligned}$$

Theorem 3.1

Let $f:(0,b]\times {\mathbb {R}} \times {\mathbb {R}} \rightarrow {\mathbb {R}}$ *be a function satisfies the condition* $(\hbox {H}_{1})$. *Then* $x\in C[-\,r,b]\cap C_{1-\gamma ;\psi }^{\gamma }[0,b]$ *is a solution of the problem* (1.1) *if and only if x satisfies the following equation*

$$\begin{aligned} x(t)=\left\{ \begin{array}{l} \frac{(\psi (t)-\psi (0))^{\gamma -1}}{\Gamma (\gamma)}\frac{1}{(1-\mu)} \sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)}\int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}} (s))ds \\ \quad +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}} (s))ds,\ \ \ \ t\in (0,b]\qquad \ \ \\ \quad \varphi (t),\qquad t\in [-\,r,0].\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \end{array} \right. \end{aligned}$$

(3.1)

Proof

According to Lemma 2.4, a solution of the problem (1.1) can be expressed by

$$\begin{aligned} x(t)&= \frac{(\psi (t)-\psi (0))^{\gamma -1}}{\Gamma (\gamma)} I_{0^{+}}^{1-\gamma ;\psi }x(0^{+}) \nonumber \\&\quad +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}} (s))ds,\ \ t\in (0,b]. \end{aligned}$$

(3.2)

Substitute $t=\tau _{i}$ into Eq. (3.2) and multiplying both sides of Eq. (3.2) by c_{i}, we get

$$\begin{aligned} c_{i}x(\tau _{i})&= c_{i}\frac{(\psi (\tau _{i})-\psi (0))^{\gamma -1}}{\Gamma (\gamma)}I_{0^{+}}^{1-\gamma ;\psi }x(0^{+})\\&\quad +\,\frac{c_{i}}{\Gamma (\alpha)} \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} I_{0^{+}}^{1-\gamma ;\psi }x(0^{+})=\sum \limits _{i=1}^{k}c_{i}x(\tau _{i}), \end{aligned}$$

which implies

\begin{aligned} I_{0^{+}}^{1-\gamma ;\psi }x(0^{+})=\frac{1}{1-\mu }\sum \limits
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_{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)}\int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi

$$(\tau_{i})-\psi(s))^{\alpha-1}f(s,x(s),x(\mathfrak{g}(s))ds.$$

(3.3)

Substituting Eq. (3.3) into Eq. (3.2), we get

$$\begin{aligned} x(t)&= \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{1-\mu}\sum\limits_{i=1}^{k}\frac{c_{i}}{\Gamma(\alpha)}\int_{0}^{\tau_{i}}\psi^{\prime}(s)(\psi(\tau_{i})-\psi(s))^{\alpha-1}f(s,x(s),x(\mathfrak{g}(s))ds \nonumber \\&\quad +\,\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\psi^{\prime}(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,x(s),x(\mathfrak{g}(s))ds,t\in(0,b]. \end{aligned}$$

(3.4)

For all $t\in[-\,r,0]$, we have $x(t)=\varphi(t)$. This means that Eq. (3.1) is satisfied.

Conversly, to prove the sufficient condition, applying $D_{0^{+}}^{\gamma;\psi}$ on both sides of Eq. (3.4), and using Lemma 2.1, we get

$$\begin{aligned} D_{0^{+}}^{\gamma;\psi}x(t)=D_{0^{+}}^{\beta(1-\alpha);\psi}f\big(t,x(t),x(\mathfrak{g}(t)\big). \end{aligned}$$

(3.5)

By the definition of $C_{1-\gamma;\psi}^{\gamma}[0,b]$ and $x\in C_{1-\gamma;\psi}^{\gamma}[0,b]$, we have $D_{0^{+}}^{\gamma;\psi}x\in C_{1-\gamma;\psi}[0,b]$. Consequently, $D_{0^{+}}^{\beta(1-\alpha);\psi}f= D^{1,\psi}I_{0^{+}}^{1-\beta(1-\alpha);\psi}f\in C_{1-\gamma;\psi}[0,b]$. For any $f\in C_{1-\gamma;\psi}[0,b]$, $t\in[0,b]$, it is obvious that $I_{0^{+}}^{1-\beta(1-\alpha);\psi}f\in C_{1-\gamma;\psi}^{1}[0,b]$. Hence *f* and $I_{0^{+}}^{1-\beta(1-\alpha);\psi}f$ satisfies the assumptions of Theorem 2.3.Now, multiplying both sides of Eq. (3.5) by $I_{0^{+}}^{\beta(1-\alpha);\psi}$, using Eq. (2.3) and Theorem 2.3, we can write

$$^{H}D_{0^{+}}^{\alpha,\beta;\psi}x(t) = f\big(t,x(t),x(\mathfrak{g}$$

$$\ldots\beta(1-\alpha);\psi}f\big$$

$(0,x(0),x({\mathfrak {g}}(0) \big)}{\Gamma (\beta (1-\alpha))}(\psi (t)-\psi (0))^{\beta (1-\alpha)-1}. \end{aligned}$

(3.6)

By Theorem 2.4, we have $I_{0^{+}}^{1-\beta (1-\alpha);\psi }f\big (0,x(0),x({\mathfrak {g}}(0)\big)=0$. Hence Eq. (3.6) reduces to

$$\begin{aligned} ^{H}D_{0^{+}}^{\alpha ,\beta ;\psi }x(t)=f\big (t,x(t),x({\mathfrak {g}}(t)\big) ,t\in (0,b]. \end{aligned}$$

Next, multiplying both sides of Eq. (3.4) by $I_{0^{+}}^{1-\gamma ;\psi }$, taking the limit as $t\rightarrow 0^{+}$ and using Theorem , we get

$$\begin{aligned} I_{0^{+}}^{1-\gamma ;\psi }x(0^{+})&= \frac{1}{1-\mu }\sum \limits _{i=1}^{k} \frac{c_{i}}{\Gamma (\alpha)}\int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds \nonumber \\&= \sum \limits _{i=1}^{k}c_{i}\frac{(\psi (\tau _{i})-\psi (0))^{\gamma -1}}{ \Gamma (\gamma)}I_{0^{+}}^{1-\gamma ;\psi }x(0^{+}) \nonumber \\&\quad +\,\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)}\int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds \nonumber \\&= \sum \limits _{i=1}^{k}c_{i}x(\tau _{i}). \end{aligned}$$

(3.7)

Thus, the nonlocal boundary condition of the problem (1.1) is satisfied. \square

Theorem 3.2

Assume that $(\hbox {H}_{1}),(\hbox {H}_{2})$ *and* $(\hbox {H}_{3})$ *hold. Then the problem* (1.1) *has a unique solution in* $C[-\,r,b]\cap C_{1-\gamma ;\psi }[0,b]$.

Proof

Define the operator ${\mathscr {G}}_{f}:C[-\,r,b]\rightarrow C[-\,r,b]$ by

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$$\begin{aligned} ({\mathscr {G}}_{f}x)(t)=\left\{ \begin{array}{l} \frac{(\psi (t)-\psi (0))^{\gamma -1}}{\Gamma (\gamma)}\frac{1}{1-\mu } \sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)}\int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds \\ \quad +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds,\ \ \ \ t\in [0,b],\qquad \ \ \\ \quad \varphi (t),\qquad t\in [-\,r,0].\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \end{array} \right. \end{aligned}$$

(3.8)

It is obvious that the operator ${\mathscr {G}}_{f}$ is well defined. Clearly, for any continuous function *f* the operator ${\mathscr{G}}_{f}$ is continuous too. Indeed, for each $t,t_{0}\in (0,b]$, we have

$$\begin{aligned}&\left| {\mathscr{G}}_{f}(x)(t)-{\mathscr{G}}_{f}(x)(t_{0})\right| \\&\quad =\left| \left[\frac{(\psi (t)-\psi (0))^{\gamma -1}}{\Gamma (\gamma)} - \frac{(\psi (t_{0})-\psi (0))^{\gamma -1}}{\Gamma (\gamma)}\right] \right. \\&\qquad \frac{1}{1-\mu }\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)} \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds \\&\qquad +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds \\&\qquad \left. -\frac{1}{\Gamma (\alpha)}\int _{0}^{t_{0}}\psi ^{\prime }(s)(\psi (t_{0})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds\right| \\&\qquad \left. \rightarrow 0,\text { as }t_{0}\rightarrow t,\right. \end{aligned}$$

and for each $t,t_{0}\in [-\,r,0]$, we get

$$\begin{aligned} \left| {\mathscr{G}}_{f}(x)(t)-{\mathscr{G}}_{f}(x)(t_{0})\right| =\left| \varphi (t)-\varphi (t_{0})\right| \rightarrow 0,\text { as } t_{0}\rightarrow t. \end{aligned}$$

Now, we prove that the operator ${\mathscr{G}}_{f}$ is a contraction maping on $C[-\,r,b]$ with respect to the norm $\left\| \cdot \right\| _{C_{1-\gamma ;\psi }[0,b]}$.

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t] , we have

$$\begin{aligned} \left| {\mathscr {G}}_{f}(x)(t)-{\mathscr {G}}_{f}(x^{*})(t)\right| =0. \end{aligned}$$

For all $t\in (0,b]$, and for each $x,x^{*}\in C\left[-\,r,b\right]$, we obtain

$$\begin{aligned} &\left| {\mathscr {G}}_{f}(x)(t)-{\mathscr {G}}_{f}(x^{*})(t)\right| \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{\gamma -1}}{\Gamma (\gamma)}\frac{1}{1-\mu }\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)}\right. \\&\qquad \left. \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}\left| f(s,x(s),x({\mathfrak {g}}(s))-f(s,x^{*}(s),x^{*}({\mathfrak {g}}(s))\right| ds\right. \\&\qquad +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left| f(s,x(s),x({\mathfrak {g}}(s))-f(s,x^{*}(s),x^{*}({\mathfrak {g}}(s))\right| ds \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{\gamma -1}}{\Gamma (\gamma)}\frac{1}{1-\mu }\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)} \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}\left(\psi (s)-\psi (0)\right) ^{\gamma -1}\right. \\&\qquad \left. \times \,{\mathscr {M}}_{f}\left(\psi (s)-\psi (0)\right) ^{1-\gamma }\left[\left| x(s)-x^{*}(s)\right| +\left| x({\mathfrak {g}} (s))-x^{*}({\mathfrak {g}}(s))\right| \right] ds\right. \\&\qquad \left. +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left(\psi (s)-\psi (0)\right) ^{\gamma -1} \right. \\&\qquad \times\,\left. {\mathscr {M}}_{f}\left(\psi (s)-\psi (0)\right) ^{1-\gamma }\left[\left| x(s)-x^{*}(s)\right| +\left| x({\mathfrak {g}} (s))-x^{*}({\mathfrak {g}}(s))\right| \right] ds\right. \\&\quad =A_{1}+A_{2.}, \end{aligned}$$

where

$$\begin{aligned}&A_{1} :=\frac{\left(\psi (t)-\psi (0)\right) ^{\gamma -1}}{\Gamma (\gamma)}\frac{1}{1-\mu }\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)} \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}\left(\psi (s)-\psi (0\right) ^{\gamma -1}\\&\quad \times \, {\mathscr {M}}_{f}\left(\psi (s)-\psi (0\right) ^{1-\gamma }\left[\left| x(s)-x^{*}(s)\right| +\left| x({\mathfrak {g}} (s))-x^{*}({\mathfrak {g}}(s))\right| \right] ds, \end{aligned}$$

and

$$\begin{aligned}&A_{2.} :=\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s) (\psi (t)-\psi (s))^{\alpha -1}\left(\psi (s)-\psi (0)\right) ^{\gamma -1}\\&\quad \times \,{\mathscr {M}}_{f}\left(\psi (s)-\psi (0)\right) ^{1-\gamma }\left[\left| x(s)-x^{*}(s)\right| +\left| x({\mathfrak {g}} (s))-x^{*}({\mathfrak {g}}(s))\right| \right] ds. \end{aligned}$$

By Theorem 2.2 we conclude that

$$\begin{aligned} A_{1}=\frac{2{\mathscr {M}}_{f}{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)} \frac{\left(\psi (t)-\psi (0)\right) ^{\gamma -1}}{\Gamma (\gamma)(1-\mu)} \sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha +\gamma -1}\left\| x-x^{*}\right\| _{C_{1-\gamma ;\psi }\left[0,b \right] }, \end{aligned}$$

(3.9)

and

$$\begin{aligned} A_{2}=\frac{2{\mathscr {M}}_{f}{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)} \left(\psi (t)-\psi (0)\right) ^{\alpha +\gamma -1}\left\| x-x^{*}\right\| _{C_{1-\gamma ;\psi }\left[0,b\right] }, \end{aligned}$$

(3.10)

where ${\mathscr {B}}(\gamma ,\alpha)$ is Beta function. From Eq. (3.9) and Eq. (3.10), it follows that

$$\begin{aligned}&\left| {\mathscr {G}}_{f}(x)(t)-{\mathscr {G}}_{f}(x^{*})(t)\right| \\&\le \frac{2{\mathscr {M}}_{f}{\mathscr {B}}(\gamma ,\alpha)\left(\psi (t)-\psi (0)\right) ^{\gamma -1}}{\Gamma (\alpha)} \\&\quad \times\,\left[\frac{1}{\Gamma (\gamma)(1-\mu)}\sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha +\gamma -1}+\left(\psi (t)-\psi (0)\right) ^{\alpha }\right] \left\| x-x^{*}\right\| _{C_{1-\gamma ;\psi }\left[0,b\right] }. \end{aligned}$$

As $0<\gamma <1$, then \frac{\left(\psi (\tau _{i})-\psi (0)\right) ^{\gamma }}{\psi (\tau

$$\begin{aligned}&\left\| {\mathscr {G}}_{f}x-{\mathscr {G}}_{f}x^{*}\right\| _{C_{1-\gamma ;\psi }\left[0,b\right] } \\&\left. \le \frac{2{\mathscr {M}}_{f}{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)}\left[\frac{1}{\Gamma (\gamma)(1-\mu)}\sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha }+\left(\psi (b)-\psi (0)\right) ^{\alpha }\right] \left\| x-x^{*}\right\| _{C_{1-\gamma ;\psi }\left[0,b\right] }.\right. \end{aligned}$$

Hence, the operator ${\mathscr {G}}_{f}$ is a contraction mapping on $C[-\,r,b]$ with respect to the norm $\left\| \cdot \right\| _{C_{1-\gamma ;\psi }[0,b]}$ due to ($\hbox {H}_{3}$). As consequence of the Banach fixed point theorem, we conclude that ${\mathscr {G}}_{f}$ has a fixed point which is a unique solution of the problem (1.1) in $C[-\,r,b]\cap C_{1-\gamma ;\psi }[0,b]$. $\square$

Theorem 3.3

Assume that ($\hbox {H}_{1}$) *and* ($\hbox {H}_{2}$) *hold. Then the problem* (1.1) *has at least one solution in* $C[-\,r,b]\cap C_{1-\gamma ;\psi }[0,b]$.

Proof

Define a bounded, closed and convex set B_{r} in $C_{1-\gamma ,\psi }[0,b]$ as follows

$$\begin{aligned} B_{r}=\left\{ x\in C_{1-\gamma ,\psi }[0,b]:\left\| x\right\| _{1-\gamma ,\psi }\le r\right\} , \end{aligned}$$

with $r\ge \frac{W}{1-\Omega }, \Omega <1$ and

$$\begin{aligned} W:=\left\| {\widetilde{f}}\right\| \frac{{\mathscr {B}}(\gamma ,\alpha)}{ \Gamma (\alpha)}\left[\frac{1}{\Gamma (\gamma)(1-\mu)} \sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha }+\left(\psi (b)-\psi (0)\right) ^{\alpha }\right] , \end{aligned}$$

where ${\widetilde{f}}(s):=\sup _{s\in [0,b]}f(s,0,0)$. We can analyze ${\mathscr {G}}_{f}(\cdot)$ as ${\mathscr {G}}_{f}x(t)=\Phi ^{1}x(t)+\Phi ^{2}x(t)$ such that

\begin{aligned} \Phi ^{1}x(t)&:= \left\{ \begin{array}{l} \frac{1}{\Gamma (\alpha)}\int

 (s),x({\mathfrak {g}}

(s))ds,\ \ t\in [0,b], \\ \varphi (t),\qquad t\in [-\,r,0]\qquad \qquad \qquad \qquad \qquad \qquad \ \ \qquad \end{array} \right. \\ \Phi ^{2}x(t)&:= \left\{ \begin{array}{l} \frac{(\psi (t)-\psi (0))^{\gamma -1}}{\Gamma (\gamma)}\frac{1}{(1-\mu)} \sum \nolimits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)} \\\times\, \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s))ds,t\in (0,b], \\ 0,\qquad t\in [-\,r,0].\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \end{array} \right. \end{aligned}

The proof was divided into several steps.

Step (1) We prove $\Phi ^{1}x+\Phi ^{2}y\in B_{r}$ for every *x*, *y* $\in B_{r}$.

For $t\in (-\,r,0]$, we have $\left| \left(\psi (t)-\psi (0)\right) ^{1-\gamma }\left(\Phi ^{1}x(t)+\Phi ^{2}y\mathfrak {(}t\mathfrak {)}\right) \right| \le \left\| \varphi \right\| _{C_{1-\gamma ,\psi }}\le r$ which implies

$$\begin{aligned} \left\| \Phi ^{1}x+\Phi ^{2}y\right\| _{C_{1-\gamma ,\psi }}\le r. \end{aligned}$$

For $t\in (0,b]$, and $x\mathfrak {,}y\in B_{r}$, we obtain

\begin{aligned}&\left| \left(\psi (t)-\psi (0)\right) ^{1-\gamma }\left(\Phi ^{1}x(t)+\Phi ^{2}y\mathfrak {(}t\mathfrak {)}\right) \right| \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{1-\gamma }}{\Gamma (\alpha)}\right. \\&\qquad \left. \int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left[\left| f(s,x(s),x({\mathfrak {g}}(s)))-f(s,0,0)\right| +\left| f(s,0,0)\right| \right] ds\right. \\&\qquad \left. +\,\frac{1}{\Gamma (\gamma)}\frac{1}{(1-\mu)}\sum \limits _{i=1}^{k} \frac{c_{i}}{\Gamma (\alpha)}\right. \\&\qquad \left. \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}\left[\left| f(s,y(s),y({\mathfrak {g}} (s)))-f(s,0,0)\right| +\left| f(s,0,0)\right| \right] ds\right. \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{1-\gamma }}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left\{ {\mathscr {M}}_{f}\left[\left| x(s)\right| +\left| x({\mathfrak {g}}(s)\right| \right] +\left| f(s,0,0)\right| \right\} ds\right. \\&\qquad +\,\frac{1}{\Gamma (\gamma)}\frac{1}{(1- [illegible])}\\&\qquad \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}\left\{ {\mathscr

{M}}_{f}\left[\left| y(s)\right| +\left| y({\mathfrak {g}}(s)\right| \right] +\left| f(s,0,0)\right| \right\} ds \\&\quad \left. \le \frac{\Gamma (\gamma)\left(\psi (t)-\psi (0)\right) ^{\alpha }}{\Gamma (\gamma +\alpha)}\left\{ 2{\mathscr {M}}_{f}\left\| x\right\| _{C_{1-\gamma ,\psi }[0,b]}+\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]}\right\} \right. \\&\qquad +\,\frac{1}{\Gamma (\alpha +\gamma)(1-\mu)}\sum \limits _{i=1}^{k}c_{i} \left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha }\left\{ 2{\mathscr {M}} _{f}\left\| y\right\| _{C_{1-\gamma ,\psi }[0,b]}+\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]}\right\} \\&\quad \left. \le \frac{{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)}\left(\psi (t)-\psi (0)\right) ^{\alpha }\left\{ 2{\mathscr {M}}_{f}r+\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]}\right\} \right. \\&\qquad +\,\frac{{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)\Gamma (\gamma)(1-\mu)}\sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha }\left\{ 2{\mathscr {M}}r+\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]}\right\} \\&\quad \left. \le \frac{2{\mathscr {M}}_{f}{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)}\left[\frac{1}{\Gamma (\gamma)(1-\mu)}\sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha }+\left(\psi (b)-\psi (0)\right) ^{\alpha }\right] r\right. \\&\qquad \left. +\,\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]} \frac{{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)}\left[\frac{1}{\Gamma (\gamma)(1-\mu)}\sum \limits _{i=1}^{k}c_{i}\left(\psi (\tau _{i})-\psi (0)\right) ^{\alpha }+\left(\psi (b)-\psi (0)\right) ^{\alpha }\right] \right. \\&\quad \left. \le \Omega r+W\le r.\right. \end{aligned}

Thus, we get \left\| \Phi ^{1}x+\Phi ^{2}y\right\| _{C_{1-\gamma ,\psi }}\le r, that is, \Phi ^{1}x+\Phi ^{2}y\in B_{r}.

Step (2) We prove \Phi ^{1}is a continuous in B_{r}. Let \{x_{n}\}_{n\ge 1}^{\infty } be a sequence in B_{r} such that x_{n}\rightarrow x\mathfrak {.}

For t\in \left(-\,r,0\right] , we have \left| \left(\psi (t)-\psi (0)\right) ^{1-\gamma }\left(\Phi ^{1}x_{n}(t)-\Phi ^{1}x(t)\right) \right| =0<r.

For t\in \left(0,b\right] , we get

 ıma }\left(\Phi ^{1}x_{n}(t)-\Phi ^{1}x(t)\right) \right| \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{1-

\gamma }}{\Gamma (\alpha)}\int \limits _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1} \right. \\&\quad \times\,\left. \left| f(s,x_{n}(s),x_{n}({\mathfrak {g}}(s)))-f(s,x(s),x({\mathfrak {g}}(s))\right| ds\right. \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{\alpha }}{\Gamma (\alpha)}\frac{\Gamma (\gamma)}{\Gamma (\alpha +\gamma)}\left\| f(\cdot ,x_{n}(\cdot),x_{n}({\mathfrak {g}}(\cdot)))-f(\cdot ,x(\cdot),x({\mathfrak {g}}(\cdot))\right\| \right. \\&\quad \left. \rightarrow 0\text { as }x_{n}\rightarrow x.\right. \end{aligned}

Step (3) We prove \Phi ^{1}is compact in B_{r}.

First, we show that \Phi ^{1} maps bounded sets into equicontinuous sets in B_{r}. For any t_{1},t_{2}\in \left(-\,r,0\right] , t_{1}<t_{2}, we have

\begin{aligned}&\left| \left(\psi (t_{2})-\psi (0)\right) ^{1-\gamma }\Phi ^{1}x(t_{2})-\left(\psi (t_{1})-\psi (0)\right) ^{1-\gamma }\Phi ^{1}x(t_{1})\right| \\&\quad \left. \le \left| \left(\psi (t_{2})-\psi (0)\right) ^{1-\gamma }\varphi (t_{2})-\left(\psi (t_{1})-\psi (0)\right) ^{1-\gamma }\varphi (t_{1})\right| \right. \\&\left. \rightarrow 0\text { as }t_{2}\rightarrow t_{1}.\right. \end{aligned}

For any t_{1},t_{2}\in \left(0,b\right] , t_{1}<t_{2} and x\in B_{r}, we get

\begin{aligned}&\left| \left(\psi (t_{2})-\psi (0)\right) ^{1-\gamma }\Phi ^{1}x(t_{2})-\left(\psi (t_{1})-\psi (0)\right) ^{1-\gamma }\Phi ^{1}x(t_{1})\right| \\&\quad \left. \le \left| \frac{\left(\psi (t_{2})-\psi (0)\right) ^{1-\gamma }}{\Gamma (\alpha)}\int _{0}^{t_{2}}\psi ^{\prime }(s)(\psi (t_{2})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s)))ds\right. \right. \\&\qquad \left. -\,\frac{\left(\psi (t_{1})-\psi (0)\right) ^{1-\gamma }}{\Gamma (\alpha)}\int _{0}^{t_{1}}\psi ^{\prime }(s)(\psi (t_{1})-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s)))ds\right| \\&\quad \left. =\frac{1}{\Gamma (\alpha)}\left| \int _{0}^{t_{1}}\psi ^{\prime }(s)\left[(\psi (t_{2})-\psi (s))^{\alpha -1}\left(\psi (t_{2})-\psi (0)\right) ^{1-\gamma }\right. \right. \right. \\&\qquad -\,\left. \left. (\psi (t_{1})-\psi (s))^{\alpha -1}\left(\psi (t_{1})-\psi (0)\right) ^{1-\gamma }\right] f(s,x(s),x({\mathfrak {g}}(s)))ds\right. \\&\qquad \left. +\,\frac{\left(\psi (t_{2})-\psi (0)\right) ^{1-\gamma }}{\Gamma (\alpha)}\int _{t_{1}}^{t_{2}}\psi ^{\prime }(s)(\psi (t_{2})-\psi (s))^{\alpha -1}\left| f(s,x(s),x({\mathfrak {g}}(s)))\right| ds\right| \\&\quad \left. \rightarrow 0\text { as }t_{2}\rightarrow t_{1}.\right. \end{aligned}

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Next, we show that $\Phi ^{1}$ is uniformly bounded on B_{r}. For $t\in \left(0,b\right]$ and $x\in B_{r}$, we have

$$\begin{aligned}&\left| \left(\psi (t)-\psi (0)\right) ^{1-\gamma }\Phi ^{1}x(t)\right| \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{1-\gamma }}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1} \right. \\&\qquad \times\,\left. \left[f(s,x(s),x({\mathfrak {g}}(s)))-f(s,0,0)+f(s,0,0)\right] ds\right. \\&\quad \left. \le \frac{\left(\psi (t)-\psi (0)\right) ^{1-\gamma }}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left[\psi (s)-\psi (0)\right] ^{\gamma -1} \right. \\&\qquad \left. \times \left[\left(\psi (s)-\psi (0)\right) ^{1-\gamma }\left| f(s,x(s),x({\mathfrak {g}}(s)))-f(s,0,0)\right| +\left| f(s,0,0)\right| \right] ds\right. \\&\quad \left. \le \frac{\Gamma (\gamma)\left(\psi (t)-\psi (0)\right) ^{\alpha }}{\Gamma (\gamma +\alpha)}\left\{ 2{\mathscr {M}}_{f}\left\| x\right\| _{C_{1-\gamma ,\psi }}+\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]}\right\} \right. \\&\quad \left. \le \frac{{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)}\left\{ 2 {\mathscr {M}}_{f}r+\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]}\right\} \left(\psi (b)-\psi (0)\right) ^{\alpha }.\right. \end{aligned}$$

It follows that

$$\begin{aligned} \left\| \Phi ^{1}x\right\| _{C_{1-\gamma ,\psi }}\le M_{0}, \end{aligned}$$

where $M_{0}:=\frac{{\mathscr {B}}(\gamma ,\alpha)}{\Gamma (\alpha)}\left\{ 2 {\mathscr {M}}_{f}r+\left\| {\widetilde{f}}\right\| _{C_{1-\gamma ,\psi }[0,b]}\right\} \left(\psi (b)-\psi (0)\right) ^{\alpha }$.

On the other hand, for $t\in \left(-\,r,0\right]$, $\varphi \in B_{r}$, we have

$$\begin{aligned} \left| \left(\psi (t)-\psi (0)\right) ^{1-\gamma }\Phi ^{1}x(t)\right| \le \left\| \varphi \right\| _{C_{1-\gamma ,\psi }}\le r. \end{aligned}$$

Thus $\Phi ^{1}$ is uniformly bounded on B_{r}. Hence by Arzelà-Ascoli theorem, $\Phi ^{1}$ is compact operator.

Step (4) We prove $\Phi ^{2}$ is a contraction mapping.

According to Theorem 3.2, the operator ${\mathscr {G}}_{f}$ is a contraction mapping on $C[-\,r,b]$ with respect to the norm $\left\| \cdot \right\| _{C_{1-\gamma ;\psi }[0,b]}$, and hence $\Phi ^{2}$ is a contraction mapping too. By the Krasnoselskii fixed point theorem there exists a fixed point *x* in B_{r}, which is a solution of the problem (1.1). $\square$

4 Ulam–Hyers–Mittag-Leffler stability

In this section, we discuss the Ulam–Hyers–Mittag-Leffler stability of the problem (1.1). The following observations are taken from [15].

Remark 4.1

A function $z\in C[-\,r,b]$ satisfies the inequality

$$\begin{aligned} \left| ^{H}D_{0^{+}}^{\alpha ,\beta ,\psi }z(t)-f(s,z(s),z({\mathfrak {g}} (s)))\right| \le \varepsilon E_{\alpha }(\psi (t)-\psi (0))^{\alpha },\ t\in (0,b], \end{aligned}$$

(4.1)

if and only if there exists a function $\eta \in C[-\,r,b]$ such that

1. $\left| \eta (t)\right| \le \varepsilon E_{\alpha }(\psi (t)-\psi (0))^{\alpha },\ \ t\in \left[-\,r,b\right] ;$
2. $^{H}D_{0^{+}}^{\alpha ,\beta ,\psi }z(t)=f\big (t,z(t),z({\mathfrak {g}}(t))\big)+\eta (t), \ t\in (0,b].$

Definition 4.1

The problem (1.1) is Ulam–Hyers–Mittag-Leffler stable with respect to $E_{\alpha }((\psi (t)-\psi (0))^{\alpha })$ if there exists $C_{E_{\alpha }}>0$ such that, for each $\varepsilon >0$ and each $z\in C[-\,r,b]$ satisfies the inequality (4.1), there exists a solution $x\in C[-\,r,b]$ of the problem (1.1) with

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$$\begin{aligned} \left| z(t)-x(t)\right| \le C_{E_{\alpha }}\varepsilon E_{\alpha }\left(\psi (t)-\psi (0)\right) ^{\alpha },\ \ t\in \left[-\,r,b\right] . \end{aligned}$$

Lemma 4.1

Let $0<\alpha <1, 0\le \beta \le 1,$ *if a function* $z\in C_{1-\gamma ,\psi }[0,b]$ *satisfies the inequality* (4.1), *then z satisfies the following integral inequality*

$$\begin{aligned}&\left| z(t)-{\mathscr {A}}_{z}-\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,z(s),z({\mathfrak {g}}(s)))ds\right| \\&\quad \left. \le \varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{\varepsilon }{\Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }(\psi (\tau _{i})-\psi (0))^{\alpha }+E_{\alpha }((\psi (t)-\psi (0))^{\alpha })\right] ,\right. \end{aligned}$$

where

$$\begin{aligned} {\mathscr {A}}_{z}:=\frac{1}{(1-\mu)}\frac{(\psi (t)-\psi (0))^{\gamma -1}}{ \Gamma (\gamma)}\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)} \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}f(s,z(s),z({\mathfrak {g}}(s)))ds. \end{aligned}$$

Proof

Indeed by Remark 4.1, we have

$$\begin{aligned} ^{H}D_{0^{+}}^{\alpha ,\beta ,\psi }z(t)=f(s,z(s),z({\mathfrak {g}}(s)))+\eta (t),\qquad t\in (0,b]. \end{aligned}$$

Then

$$\begin{aligned} z(t)&= {\mathscr {A}}_{z}-\frac{1}{(1-\mu)}\frac{\left(\psi (t)-\psi (0)\right) ^{\gamma -1}}{\Gamma (\gamma)}\sum \limits _{i=1}^{k}\frac{c_{i}}{ \Gamma (\alpha)}\int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}\eta (s)ds \\&\quad +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,z(s),z({\mathfrak {g}}(s)))ds \\&\quad [illegible])(\psi (t)-\psi (s))^{\alpha -1}\eta (s)ds. \end{aligned}$$

It follows that

$$\begin{aligned}&\left| z(t)-{\mathscr {A}}_{z}-\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,z(s),z({\mathfrak {g}}(s)))ds\right| \\&\quad \left. \le \left| \frac{1}{(1-\mu)}\right| \frac{1}{\Gamma (\gamma)}\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)}\int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}\left| \eta (s)\right| ds\right. \\&\qquad \left. +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left| \eta (s)\right| ds\right. \\&\quad \left. \le \left| \frac{1}{(1-\mu)}\right| \frac{\varepsilon }{ \Gamma (\gamma)}\sum \limits _{i=1}^{k}\frac{c_{i}}{\Gamma (\alpha)} \int _{0}^{\tau _{i}}\psi ^{\prime }(s)(\psi (\tau _{i})-\psi (s))^{\alpha -1}E_{\alpha }(\psi (s)-\psi (0))^{\alpha }ds\right. \\&\qquad \left. +\,\frac{\varepsilon }{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}E_{\alpha }(\psi (s)-\psi (0))^{\alpha }ds.\right. \end{aligned}$$

By definition of Mittag-Leffler function and Theorem 2.2, we get

$$\begin{aligned}&\left| z(t)-{\mathscr {A}}_{z}-\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,z(s),z({\mathfrak {g}}(s)))ds\right| \\&\quad \left. \le \left| \frac{1}{(1-\mu)}\right| \frac{\varepsilon }{ \Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}\sum \limits _{k=0}^{\infty }\frac{ (\psi (\tau _{i})-\psi (0)^{\alpha })^{k+1}}{\Gamma ((k+1)\alpha +1)} +\varepsilon \sum \limits _{k=0}^{\infty }\frac{(\psi (t)-\psi (0)^{\alpha })^{k+1}}{\Gamma ((k+1)\alpha +1)}\right. \\&\quad \left. \le \varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{1}{\Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}\sum \limits _{n=0}^{ \infty }\frac{(\psi (\tau _{i})-\psi (0)^{\alpha })^{n}}{\Gamma (n\alpha +1)} +\sum \limits _{k=0}^{\infty }\frac{(\psi (t)-\psi (0)^{\alpha })^{n}}{\Gamma (n\alpha +1)}\right] \right. \\&\quad \left. =\varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{1 }{\Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (t)-\psi (0))^{\alpha })\right] .\right. \end{aligned}$$

$\square$

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In the forthcoming theorem, we prove the stability results for the problem (1.1).

Theorem 4.1

Assume that (H_{1}) *and* (H_{2}), *are satisfied. If*

$$\begin{aligned} 2\frac{{\mathscr {B}}(\gamma ,\alpha){\mathscr {M}}_{f}}{\Gamma (\alpha)}(\psi (t)-\psi (0))^{\alpha }<1. \end{aligned}$$

(4.2)

Then

$$\begin{aligned} ^{H}D_{0^{+}}^{\alpha ,\beta ,\psi }x(t)=f(s,x(s),x({\mathfrak {g}}(s))),\ \ t\in (0,b], \end{aligned}$$

(4.3)

is Ulam–Hyers–Mittag-Leffler stable.

Proof

Let $\varepsilon >0$, $z\in C[-\,r,b]\cap C_{1-\gamma ;\psi }\left[0,b\right]$ be a function satisfying the inequality

$$\begin{aligned} \left| ^{H}D_{0^{+}}^{\alpha ,\beta ,\psi }z(t)-f(s,z(s),z({\mathfrak {g}} (s)))\right| \le \varepsilon E_{\alpha }(\psi (t)-\psi (0))^{\alpha },\ t\in (0,b], \end{aligned}$$

(4.4)

and let $x\in C[-\,r,b]\cap C_{1-\gamma ;\psi }\left[0,b\right]$ be the unique solution of the following problem

$$\begin{aligned} \left\{ \begin{array}{l} ^{H}D_{0^{+}}^{\alpha ,\beta ;\psi }x(t)=f(t,x(t),x({\mathfrak {g}}(t)), t\in (0,b] \\ I_{0^{+}}^{1-\gamma ;\psi }x(0^{+})=I_{0^{+}}^{1-\gamma ;\psi }z(0^{+}),\qquad \qquad \qquad \\ x(t)=z(t),\ \ t\in [-\,r,0].\ \end{array} \right. \end{aligned}$$

Now, by using Theorem 3.2, we have

$$\begin{aligned} x(t)=\left\{ \begin{array}{l} {\mathscr {A}}_{x}+\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s)))ds, t\in \left(0,b\right] , \\ z(t),\qquad \ t\in \left[-\,r,0\right] ,\qquad \qquad \qquad \qquad \qquad \qquad \end{array} \right. \end{aligned}$$

Note that, for all $\ t\in [-\,r,0], \left| z(t)-x(t)\right| =0$. Since $I_{0^{+}}^{1-\gamma ;\psi }x(0^{+})=I_{0^{+}}^{1-\gamma ;\psi }z(0^{+})$, we can easily prove that ${\mathscr {A}}_{x}={\mathscr {A}}_{z}$. Hence, from ($\hbox {H}_{2}$) and Lemma 4.1, then for each $t\in \left(0,b \right]$, we have

$$\begin{aligned} \left| z(t)-x(t)\right|&\le \left| z(t)-{\mathscr {A}}_{z}- \frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,z(s),z({\mathfrak {g}}(s)))ds\right| \nonumber \\&\quad +\,\left| \frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,z(s),z({\mathfrak {g}}(s)))ds\right. \nonumber \\&\quad \left. -\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}f(s,x(s),x({\mathfrak {g}}(s)))ds\right| \nonumber \\&\le \varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{1}{ \Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (t)-\psi (0))^{\alpha })\right] \nonumber \\&\quad +\,\frac{1}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left| f(s,z(s),z({\mathfrak {g}}(s)))-f(s,x(s),x(\mathfrak { g}(s)))\right| ds \nonumber \\&\le \varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{1}{ \Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (t)-\psi (0))^{\alpha })\right] \nonumber \\&\quad +\,\frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left[\left| z(s)-x(s)\right| +\left| z({\mathfrak {g}}(s))-x({\mathfrak {g}}(s))\right| \right] ds. \end{aligned}$$

(4.5)

Now, for each $y\in C([-\,r,b], {\mathbb {R}} ^{+})$, we define an operator ${\mathscr {U}}:C([-\,r,b], {\mathbb {R}} ^{+})\rightarrow C([-\,r,b], {\mathbb {R}} ^{+})$ by

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$$\begin{aligned} {\mathscr {U}}y(t)=\left\{ \begin{array}{l} \varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{1}{\Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (t)-\psi (0))^{\alpha })\right] \\ +\frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}y(s)ds\\ +\frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}y({\mathfrak {g}}(s))ds, t\in \left(0,b \right] ,\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \\ 0,\qquad \ t\in \left[-\,r,0\right] .\qquad \qquad \qquad \qquad \ \ \ \ \ \ \ \ \ \ \qquad \qquad \qquad \qquad \end{array} \right. \end{aligned}$$

We prove that ${\mathscr {U}}$ is a Picard operator. Using ($\hbox {H}_{2}$), for $t\in \left(0,b\right]$ and $y_{1},y_{2}\in C([-\,r,b], {\mathbb {R}} ^{+})$, we have

$$\begin{aligned} \left| {\mathscr {U}}y_{1}(t)-{\mathscr {U}}y_{2}(t)\right|& \le \frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left| y_{1}(s)-y_{2}(s)\right| ds \\&\quad +\,\frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}\left| y_{1}({\mathfrak {g}}(s))-y_{2}({\mathfrak {g}}(s))\right| ds \\&\le \frac{2\Gamma (\gamma){\mathscr {M}}_{f}}{\Gamma (\gamma +\alpha)} (\psi (t)-\psi (0))^{\gamma -1+\alpha }\left\| y_{1}-y_{2}\right\| _{C_{1-\gamma ,\psi }[0,b]} \\&= 2\frac{{\mathscr {B}}(\gamma ,\alpha){\mathscr {M}}_{f}}{\Gamma (\alpha)} (\psi (t)-\psi (0))^{\gamma -1+\alpha }\left\| y_{1}-y_{2}\right\| _{C_{1-\gamma ,\psi }[0,b]}, \end{aligned}$$

which implies

$$\begin{aligned} \left\| {\mathscr {U}}y_{1}-{\mathscr {U}}y_{2}\right\| _{C_{1-\gamma ,\psi [0,b]}}\le 2\frac{{\mathscr {B}}(\gamma ,\alpha){\mathscr {M}}_{f}}{\Gamma (\alpha)}(\psi (t)-\psi (0))^{\alpha }\left\| y_{1}-y_{2}\right\| _{C_{1-\gamma ,\psi }[0,b]}. \end{aligned}$$

By Eq. (4.2) we conclude that ${\mathscr {U}}$ is a contraction mapping on $C[-\,r,b]$ with respect to the norm $\left\| \cdot \right\| _{C_{1-\gamma ;\psi }[0,b]}$.

According to Banach fixed point theorem, we deduce that ${\mathscr {U}}$ is a Picard

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b], we have

$$\begin{aligned} y^{*}(t)&= \left({\mathscr {U}}y^{*}\right) (t) \nonumber \\&= \varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{1}{ \Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha } ((\psi (t)-\psi (0))^{\alpha })\right] \nonumber \\&\quad +\,\frac{{\mathscr {M}}_{f}} {\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}y^{*} (s)ds \nonumber \\&\quad +\,\frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi (t)-\psi (s))^{\alpha -1}y^{*}({\mathfrak {g}} (s))ds\text {.} \end{aligned}$$

(4.6)

Next, we prove that the solution y^{*} is increasing. Let $\sigma :=\min _{s\in \left(0,b\right] }[y^{*}(s)+y^{*}(h(s))]\in {\mathbb {R}} _{+}$, then for all $0\le t_{1} <t_{2}\le b$, we have

$$\begin{aligned}&y^{*}(t_{2})-y^{*}(t_{1}) \\&\quad \left. =\varepsilon \left[E_{\alpha }((\psi (t_{2})-\psi (0))^{\alpha })-E_{\alpha }((\psi (t_{1})-\psi (0))^{\alpha })\right] \right. \\&\qquad \left. +\,\frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t_{1}}\psi ^{\prime }(s)\left[(\psi (t_{1})-\psi (s))^{\alpha -1}-(\psi (t_{2})-\psi (s))^{\alpha -1}\right] \left[y^{*}(s)+y^{*}({\mathfrak {g}}(s))\right] ds\right. \\&\qquad +\,\frac{{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{t_{1}}^{t_{2}}\psi ^{\prime }(s)(\psi (t_{2})-\psi (s))^{\alpha -1}\left[y^{*}(s)+y^{*}({\mathfrak {g}} (s))\right] ds \\&\quad \left. \ge \varepsilon \left[E_{\alpha }((\psi (t_{2})-\psi (0))^{\alpha })-E_{\alpha }((\psi (t_{1})-\psi (0))^{\alpha })\right] \right. \\&\qquad \left. +\,\frac{{\mathscr {M}}_{f}\sigma }{\Gamma (\alpha +1)}\left[(\psi (t_{2})-\psi (0))^{\alpha }-(\psi (t_{1})-\psi (0))^{\alpha }+(\psi (t_{2})-\psi (t_{1}))^{\alpha }\right] \right. \\&\quad \left. >0.\right. \end{aligned}$$

Therefore y^{*} is increasing, and hence $y^{*}({\mathfrak {g}} (t))\le y^{*}(t)$ due to ${\mathfrak {g}}(t)\le t$, then Eq. (4.6) becomes

$$\begin{aligned} y^{*}(t)&\le \varepsilon \left[\left| \frac{1}{(1-\mu)} \right| \frac{1} {\Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (t)-\psi (0))^{\alpha })\right] \\&\quad +\,\frac{2{\mathscr {M}}_{f}}{\Gamma (\alpha)}\int _{0}^{t}\psi ^{\prime }(s)(\psi$$

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Using Lemma 2.3, we obtain

$$\begin{aligned} y^{*}(t)&\le \varepsilon \left[\left| \frac{1}{(1-\mu)} \right| \frac{1}{\Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (t)-\psi (0))^{\alpha })\right] \\&\quad \times \,E_{\alpha }(2{\mathscr {M}}_{f}\left(\psi (t)-\psi (0)\right) ^{\alpha }) \\&\le \varepsilon \left[\left| \frac{1}{(1-\mu)}\right| \frac{1}{ \Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (b)-\psi (0))^{\alpha })\right] \\&\quad \times \,E_{\alpha }(2{\mathscr {M}}_{f}\left(\psi (t)-\psi (0)\right) ^{\alpha }). \end{aligned}$$

Take $C_{E_{\alpha }}=\left(\left| \frac{1}{(1-\mu)}\right| \frac{1 }{\Gamma (\gamma)}\sum \limits _{i=1}^{k}c_{i}E_{\alpha }((\psi (\tau _{i})-\psi (0))^{\alpha })+E_{\alpha }((\psi (b)-\psi (0))^{\alpha })\right)$, we get

$$\begin{aligned} y^{*}(t)\le C_{E_{\alpha }}\varepsilon E_{\alpha }\left(\psi (t)-\psi (0)\right) ^{\alpha }. \end{aligned}$$

In particular, if $y=|z-x|$, from inequality (4.5), $y\le {\mathscr {U}} y^{*}$ and applying the Lemma 2.2, we obtain $y\le y^{*}$, where ${\mathscr {U}}$ is an increasing Picard operator. As a result, we get

$$\begin{aligned} \left| z(t)-x(t)\right| \le C_{E_{\alpha }}\varepsilon E_{\alpha }\left(\psi (t)-\psi (0)\right) ^{\alpha },\qquad \ t\in \left[-\,r,b\right] . \end{aligned}$$

Thus $^{H}D_{0^{+}}^{\alpha ,\beta ,\psi }x(t)=f(s,x(s),x({\mathfrak {g}} (s))),\ \ t\in (0,b]$ is Ulam–Hyers–Mittag-Leffler stable. $\square$

5 Examples

In this section will give two examples to illustrate our results.

Example 5.1

Consider the following problem

$$\begin{aligned} \left\{ \begin{array}{l} ^{H}D_{0^{+}}^{\frac{1}{3},\frac{2}{3};2^{t}}x(t)=\frac{1}{8}\left[\tan ^{-1}(x(t))+\frac{x^{2}(t-1)}{1+x^{2}(t-1)}\right] ,t\in \left(0,1 \right] , \\ I_{0^{+}}^{\frac{5}{9};2^{t}}\left[x(0)\right] =\frac{2}{3}x(\frac{1}{2}),\qquad \qquad \ \ \ \ \ \ \\ x(t)=e^{t},\ t\in \left[-1,0\right] ,\ \ \ \ \ \ \ \qquad \end{array} \right. \end{aligned}$$

(5.1)

here $\alpha =\frac{1}{3},\beta =\frac{2}{3}, \gamma =\alpha +\beta -\alpha \beta =\frac{4}{9},k=1,\tau _{1}=\frac{1}{2},c_{1}=\frac{2}{3}$, $(0,b]=(0,1]$, $\psi (t)=2^{t.}$, ${\mathfrak {g}}(t)=t-1$, and $f(t,x(t),x({\mathfrak {g}}(t))=\frac{1}{8}\left[\tan ^{-1}(x(t))+\frac{x^{2}(t-1)}{ 1+x^{2}(t-1)}\right]$. Thus, for all $x,x^{*}\in {\mathbb {R}}^{+}$ and $t\in (0,1]$, we have

$$\begin{aligned} \left| f(t,x(t),x({\mathfrak {g}}(t)-f(t,x^{*}(t),x^{*}({\mathfrak {g}}(t))\right| \le \frac{1}{8}\left| x(t)-x^{*}(t)\right| +\frac{1}{8}\left| x({\mathfrak {g}}(t))-x^{*}({\mathfrak {g}} (t))\right| , \end{aligned}$$

Clearly, the conditions (H_{1}) and (H_{2}) hold with ${\mathscr {M}}_{f}= \frac{1}{8}$, It is easy to check that the inequality in (H_{3}) also holds. Indeed, by some simple calculations, we get $\mu \simeq 0.91$ and $\Omega \simeq 0.65<1$.

Now, all the hypotheses in Theorem 3.2 are satisfied, so the problem (5.1) has a unique solution in $C[-\,1,1]\cap C_{\frac{5}{9} ;2^{t.}}[0,1]$. Finally, we see that the inequality

$$\begin{aligned} \left| ^{H}D_{0^{+}}^{\frac{1}{3},\frac{2}{3} ;2^{t}}z(t)-f(t,z(t),z(t-1))\right| \le \varepsilon E_{\frac{1}{3} }((2^{t}-1)^{\frac{1}{3}}) \end{aligned}$$

is satisfied. Then the Eq. (4.3) is Ulam–Hyers–Mittag-Leffler stable with

$$\begin{aligned} \left| z(t)-x(t)\right| \mid \le C_{E_{\frac{1}{3}}}\ \varepsilon E_{\frac{1}{3}}((2^{t}-1)^{\frac{1}{3}}),\ \ t\in \left[-1,1\right] , \end{aligned}$$

where

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$$\begin{aligned} C_{E_{\frac{1}{3}}}=\left(\frac{1}{0.09}\frac{\frac{1}{4}}{\Gamma (\frac{4}{ 9})}\frac{2}{3}E_{\frac{1}{3}}(2^{\frac{1}{2}}-1)^{\frac{1}{3}}+E_{\frac{1}{3 }}(1)^{\frac{1}{3}}\right) \simeq 11>0 \end{aligned}$$

Note that,

$$\begin{aligned} E_{\frac{1}{n}}(z)=e^{z^{n}}\left[1+n\int _{0}^{t}e^{-x^{n}}\left(\sum _{k=1}^{n-1}\frac{x^{k-1}}{\Gamma (\frac{k}{n})}\right) dx\right] ,(n\in {\mathbb {N}} \backslash \{1\}). \end{aligned}$$

Example 5.2

Consider the following problem

$$\begin{aligned} \left\{ \begin{array}{l} ^{H}D_{0^{+}}^{\frac{1}{2},\frac{1}{3};2^{t}}x(t)=\frac{1}{4}\left[\cos (2x(t-1))+\frac{x^{2}(t)}{1+x^{2}(t)}\right] ,t\in \left(0,\frac{1 }{4}\right] , \\ I_{0^{+}}^{\frac{1}{3};2^{t}}\left[x(0)\right] =\frac{1}{6}x(\frac{1}{5})+ \frac{1}{5}x(\frac{1}{4}),\qquad (i=1,2) \\ x(t)=\sin (t),\ t\in \left[-1,0\right] ,\qquad \qquad \ \ \qquad \end{array} \right. \end{aligned}$$

(5.2)

here $\alpha =\frac{1}{2},\beta =\frac{1}{3}, \gamma =\alpha +\beta -\alpha \beta =\frac{2}{3},k=2,\tau _{1}=\frac{1}{5},\tau _{2}=\frac{1}{4} ,c_{1}=\frac{1}{6},c_{2}=\frac{1}{5}, (0,b]=(0,\frac{1}{4}], \psi (t)=2^{t.}, {\mathfrak {g}}(t)=t-1$, and $f(t,x(t),x({\mathfrak {g}}(t))=\frac{1 }{4}\left[\cos (2x(t-1))+\frac{x^{2}(t)}{1+x^{2}(t)}\right]$. Thus, for all $x,x^{*}\in {\mathbb {R}} ^{+}$ and $t\in (0,\frac{1}{4}]$, we have

$$\begin{aligned} \left| f(t,x(t),x({\mathfrak {g}}(t)-f(t,x^{*}(t),x^{*}({\mathfrak {g}}(t))\right| \le \frac{1}{4}\left| x(t)-x^{*}(t)\right| +\frac{1}{4}\left| x({\mathfrak {g}}(t))-x^{*}({\mathfrak {g}} (t))\right| , \end{aligned}$$

Clearly, the conditions (H_{1}) and (H_{2}) hold with ${\mathscr {M}}_{f}= \frac{1}{4}$, It is easy to check that the inequality in (H_{3})

also holds. Indeed, by some simple calculations, we get $\mu \simeq 0.49$ and $\Omega \simeq 0.49<1$.

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Since, all the hypotheses in Theorem 3.2 are satisfied, the problem (5.2) has a unique solution in $C[-\,1,\frac{1}{4}]\cap C_{\frac{1}{2} ;2^{t.}}[0,\frac{1}{4}]$. Finally, we see that the inequality

$$\begin{aligned} \left| ^{H}D_{0^{+}}^{\frac{1}{2},\frac{1}{3} ;2^{t}}z(t)-f(t,z(t),z(t-1))\right| \le \varepsilon E_{\frac{1}{2} }((2^{t}-1)^{\frac{1}{2}}) \end{aligned}$$

is satisfied. Then the Eq. (4.3) is Ulam–Hyers–Mittag-Leffler stable with

$$\begin{aligned} \left| z(t)-x(t)\right| \mid \le C_{E_{\frac{1}{2}}}\ \varepsilon E_{\frac{1}{2}}((2^{t}-1)^{\frac{1}{2}}),\ \ t\in \left[-1,\frac{1}{4} \right] , \end{aligned}$$

where

$$\begin{aligned} C_{E_{\frac{1}{2}}}=\left(\frac{1}{(1-0.49)}\frac{\frac{1}{2}}{\Gamma (\frac{2}{3})}\frac{2}{3}E_{\frac{1}{2}}\left(2^{\frac{1}{5}}-1\right) ^{\frac{1}{2}}+\frac{3}{2}E_{\frac{1}{2}}\left(2^{\frac{1}{4}}-1\right) ^{ \frac{1}{2}}\right) \simeq 2.67>0. \end{aligned}$$

6 Conclusion

We have obtained existence, uniqueness and Ulam–Hyers–Mittag-Leffler stability results for the solution of nonlocal Cauchy problem for ψ -Hilfer fractional functional differential equations based on the reduction of fractional differential equations to integral equations. We employed the Picard operator method, fixed point theorems and generalized Gronwall's inequality to obtain our results. We trust the reported results here will have a positive impact on the development of further applications in engineering and applied sciences.

References

1. Abbas, S., Benchohram, M., Graef, J.R.: Implicit Fractional Differential and Integral Equations: Existence and Stability. WalterdeGruyterGmbH, Berlin (2018)

Book Google Scholar

2. Abdo, M.S., Panchal, S.K.: Fractional integro-differential equations involving \psi -Hilfer fractional derivative. Adv. Appl. Math. Mech. **11**, 338–359 (2019)

Article MathSciNet Google Scholar

3. Abdo, M.S., Panchal, S.K., Shafei Hussien, H.: Fractional integro-differential equations with nonlocal conditions and \psi -Hilfer fractional derivative. Math. Model. Anal. **24**(4), 564–584 (2019). https://doi.org/10.3846/mma.2019.034

Article MathSciNet Google Scholar

4. Agrawal, O.P.: Some generalized fractional calculus operators and their applications in integral equations. Fract. Calc. Appl. Anal. **15**, 700–711 (2012)

Article MathSciNet Google Scholar

5. Arara, A., Benchohra, M., Hamidi, N., Nieto, J.J.: Fractional order differential equations on an unbounded domain. Nonlinear Anal. Theory Methods Appl. **72**, 580–586 (2010)

Article MathSciNet Google Scholar

6. Burton, T.A.: A fixed-point theorem of Krasnoselskii. Appl. Math. Lett. **11**(1), 85–88 (1998)

Article MathSciNet Google Scholar

7. Furati, K.M., Kassim, M.D.: Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput. Math. Appl. **64**, 1616–1626 (2012)

Article MathSciNet Google Scholar

8. Hilfer, R.: Application of Fractional Calculus in Physics. World Scientific, Singapore (1999)

Google Scholar

9. Hyers, D.H., Isac, G., Rassias, Th.M.: Stability of functional equations in several variables. In: Progress in Nonlinear Differential Equations and Their Applications, vol. 34, Birkhauser, Boston (1998)

10. Goodrich, C.S.: Existence of a positive solution to a class of fractional differential equations. Appl. Math. Lett. **23**, 1050–1055 (2010)

Article MathSciNet Google Scholar

11. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Functions, Related Topics and Applications. Springer, Berlin (2014)

Book Google Scholar

12. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, vol. 207. Elsevier, Amsterdam (2006)

MATH Google Scholar

13. Kucche, K.D., Shikhare, P.U.: Ulam–Hyers stability of integro differential equations in banach spaces via Pachpatte's inequality. Asian Eur J. Math. **11**(04), 1850062 (2018)

Article MathSciNet Google Scholar

14. Kucche, K.D., Mali, A.D., Sousa, J.V.D.C.: On the nonlinear \psi -Hilfer fractional differential equations. Comput. Appl. Math. **38**(2), 73 (2019)

Article MathSciNet Google Scholar

15. Liu, K., Wang, J., O'Regan, D.: Ulam–Hyers–Mittag-Leffler stability for \psi -Hilfer fractional-order delay differential equations. Adv. Differ. Equ. **50**, 1–12 (2019)

MathSciNet MATH Google Scholar

16. Oliveira, E.C., Sousa, J.V.D.C.: Ulam–Hyers–Rassias stability for a class of fractional integro-differential equations. Results Math. **73**, 111 (2018)

Article MathSciNet Google Scholar

17. Otrocol, D., Ilea, V.: Ulam stability for a delay differential equation. Cent. Eur. J. Math. **11**, 1296–1303 (2013)

MathSciNet MATH Google Scholar

18. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations to Methods of Their Solution and Some of Their Applications, Mathematics in Science and Engineering, vol. 198. Elsevier, Amsterdam (1999)

MATH Google Scholar

19. Rassias, T.M.: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**(2), 297–300 (1978)

Article MathSciNet Google Scholar

20. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1987)

MATH Google Scholar

21. Shah, K., Ali, A., Bushnaq, S.: Hyers–Ulam stability analysis to implicit Cauchy problem of fractional differential equations with impulsive conditions. Math. Methods Appl. Sci. **41**, 8329–8343 (2018)

Article MathSciNet Google Scholar

22. Sousa, J.V.D.C., de Oliveira, E.C.: On the \psi -Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. **60**, 72–91 (2018)

Article MathSciNet Google Scholar

23. Sousa, J.V.D.C., Kucche, J.K.D., de Oliveira, E.C.: Stability of \psi -Hilfer impulsive fractional differential equations. Appl. Math. Lett. **88**, 73–80 (2019)

Article MathSciNet Google Scholar

24. Sousa, J.V.D.C., de Oliveira, E.C.: A Gronwall inequality and the Cauchy-type problem by means of \psi -Hilfer operator. Differ. Equ. Appl. **11**(1), 87–106 (2019)

MathSciNet MATH Google Scholar

25. Sousa, J.V.D.C., de Oliveira, E.C.: On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the \psi -Hilfer operator. Fixed Point Theory Appl. **20**, Article ID 96 (2018)

26. Sousa, J.V.D.C., de Oliveira, E.C.: Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation. Appl. Math. Lett. **81**, 50–56 (2018)

Article MathSciNet Google Scholar

27. Sousa, J.V.D.C., de Oliveira, E.C.: Leibniz type rule: \psi -Hilfer fractional operator. Commun. Nonlinear Sci. Numer. Simul. **77**, 305–311 (2019)

Article MathSciNet Google Scholar

28. Teodoro, G.S., Machado, J.T., De Oliveira, E.C.: A review of definitions of fractional derivatives and other operators. J. Comput. Phys. **388**, 195–208 (2019)

Article MathSciNet Google Scholar

29. Ulam, S.M.: Problems in Modern Mathematics (Chapter 6). Wiley, New York (1960)

Google Scholar

30. Ulam, S.M.: A Collection of mathematical problems. In: Interscience Tracts in Pure and Applied Mathematics, vol. 8, Inter-science, New York (1960)

31. Wang, J., Zhang, Y.: Ulam–Hyers–Mittag-Leffler stability of fractional-order delay differential equations. Optimization **63**, 1181–1190 (2014)

Article MathSciNet Google Scholar

32. Wang, J., Zhou, Y., Feĉkan, Y.M.: Nonlinear impulsive problems for fractional differential equations and Ulam stability. Comput. Math. Appl. **64**, 3389–3405 (2012)

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