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Existence and Ulam–Hyers stability results of a coupled system of ψ-Hilfer sequential fractional differential equations

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Abstract

The major goal of this work is to investigate sufficient conditions of existence, uniqueness, and Ulam–Hyers stability of solutions for a coupled system of ψ-Hilfer sequential fractional differential equations with two-point boundary conditions. Some standard fixed point theorems of Leray–Schauder alternative and Banach are applied to establish these results. A pertinent example is provided to corroboration the results obtained.

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MSC

34A08; 34B15; 34A12; 47H10

Keywords

, ψ.-Hilfer sequential FDEs; Boundary conditions; Coupled system; Existence; Fixed point theorem

1. Introduction

Fractional differential equations (FDEs) have acquired considerable significance because it has varied applications in numerous applied sciences and engineering [1], [2], [3]. There are various definitions of fractional calculus (FC) that developed the FDEs whether in modeling or descript the memory accurately. Among these famous operators are Riemann–Liouville (RL) (1832), Riemann (1849), Grünwald Letnikov (1867), Caputo (1997), Hilfer (2000), as well as Hadamard (1891) which are the most used.

The FC over time has been acquisition increasing significance in scientific research [1], [2], [3], [4], [5]. With novel definitions of FC [6], [7], [8], [9], [10], [11], [12] there have appeared some applications in a few areas of study [13], [14], [15], [16]. There is a prominent and noticeable interest in the investigation of qualitative characteristics of solutions (existence, uniqueness, stability) of FDEs. Recently, Sousa etal. and Abdo etal. in the papers series [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27] studied the existence, uniqueness, continuous dependence, and different types of stability in Ulam–Hyers sense for various kinds of FDEs.

On the other side, the study of coupled systems involving FDEs is also important as such systems occur in various problems of applied nature, for instance, see [28], [29], [30], [31], [32]. For some theoretical works on coupled systems of FDEs, we refer to series of papers [33], [34], [35], [36], [37], [38], [39]. Very recently, Abbas etal. [40] studied the existence and uniqueness of solution for the following fractional coupled system

$$\begin{cases} \left(D_{0^+}^{\alpha_1,\beta_1} u\right)(\varrho, w) = f_1\left(\varrho, u\left(\varrho, w\right), v\left(\varrho, w\right), w\right), \varrho \in [0,\chi], \\ \left(D_{0^+}^{\alpha_2,\beta_2} v\right)(\varrho, w) = f_2\left(\varrho, u\left(\varrho, w\right), v\left(\varrho, w\right), w\right), \quad w \in \Omega, \end{cases}$$

with coupled conditions

$$\left(I_{0^+}^{1-\gamma_1} u\right)(0, w) = \phi_1\left(w\right) \text{ and } \left(I_{0^+}^{1-\gamma_2} v\right)(0, w) = \phi_2\left(w\right),$$

where $\chi > 0$, $\alpha_i \in (0,1)$, $\beta_i \in [0,1]$, (Ω, A) is a measurable space, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, $\phi_i : \Omega \to \mathbb{R}^m$, $f_i : [0,\chi] \times \mathbb{R}^m \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$; $i = 1, 2$, are given functions, $I_{0^+}^{1-\gamma_i}$ is the left-sided mixed RL integral of order $1-\gamma_i$, and $D_{0^+}^{\alpha_i,\beta_i}$ is the Hilfer operator of order α_i and type β_i, $i = 1, 2$. Saengthong etal. [41] considered a system of Hilfer–Hadamard sequential FDEs

$$\begin{cases} \left({}_HD_{1^+}^{\alpha_1,\beta_1} + k_1 \quad {}_HD_{1^+}^{\alpha_1-1,\beta_1}\right)u(\varrho) = f(\varrho, u(\varrho), v(\varrho)), \quad \varrho \in [1, e], \\ \left({}_HD_{1^+}^{\alpha_2,\beta_2} + k_1 \quad {}_HD_{1^+}^{\alpha_1-1,\beta_1}\right)v(\varrho) = g(\varrho, u(\varrho), v(\varrho)), \quad \varrho \in [1, e], \\ u_1(1) = 0, \quad u_1(e) = A_1, \\ u_2(1) = 0, \quad u_2(e) = A_2, \end{cases} \tag{1.1}$$

The authors obtained some existence and uniqueness results on system (1.1) via standard fixed point theorems.

Motivated by preceding works above, in current paper, we investigate some existence, uniqueness and Ulam–Hyers stability results for more general coupled system of ψ-Hilfer sequential FDEs:

$$\begin{cases} D_{a^+}^{\alpha_1,\beta_1,\psi}u_1(\varrho) + L_1 D_{a^+}^{\alpha_1-1,\beta_1,\psi}u_1(\varrho) = f_1(\varrho, u_1(\varrho), u_2(\varrho)), \\ \quad \varrho \in \mathscr{J} = (a, \chi], \chi > a, \\ D_{a^+}^{\alpha_2,\beta_2,\psi}u_2(\varrho) + L_2 D_{a^+}^{\alpha_2-1,\beta_1,\psi}u_2(\varrho) = f_2(\varrho, u_1(\varrho), u_2(\varrho)), \\ u_1(a) = 0, \quad u_1(\chi) = Q_1, \\ u_2(a) = 0, \quad u_2(\chi) = Q_2, \end{cases} \tag{1.2}$$

where $D_{a^+}^{\alpha_i,\beta_i,\psi}$ denotes the ψ-Hilfer fractional derivative of order $\alpha_i \in (1, 2)$, and type $\beta_i \in [0, 1]$, $L_i, Q_i \in \mathbb{R}_+$, $(i = 1, 2)$, $f_i : \mathscr{J} \times \mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{X}$, $(i = 1, 2)$ are continuous fulfilling some suppositions that will be described later, and $\mathscr{X}$ is the Banach space.

The coupled system of ψ-Hilfer sequential FDEs with boundary conditions considered in the current work is the more wide class of coupled system of BVPs that incorporates the BVP for sequential FDEs involving the most broadly used Riemann–Liouville and Caputo fractional derivatives. Regardless of this, the coupled system (1.2) for various values of a function ψ and parameter β_i $(i = 1, 2)$ includes the study of coupled system of sequential FDEs involving the Hilfer, Hadamard, Katugampola, and many other fractional derivatives operators. Moreover, the results gained in our paper includes the results of Ahmad et al.[42], Abbas[43] and Saengthong et al.[41], also, it will be a useful contribution to the existing literature on the topic.

The major contribution of the current paper is to obtain an equivalent fractional integral system and to establish the existence, uniqueness and Ulam–Hyers stability of solutions for a coupled system of ψ-Hilfer sequential FDEs with minimal conditions of nonlinear terms. Our analysis relying on fixed point theorems of Banach and Leray–Schauder. Though we make use of the standard methodology to obtain our results, yet its exposition to the proposed system is new.

This paper is systematized as follows: In Section2, we rendering the rudimentary definitions and prove some lemmas that are applied throughout this paper, also we present the concepts of some fixed point theorems. In Section3, we prove the existence, uniqueness and Ulam–Hyers stability of solutions for coupled system (1.2). In Section5, we give a pertinent example to illustrate our results. Concluding remarks about our results in the last section.

2. Preliminar

In this part, we give important definitions and auxiliary lemmas that have pertinent to our main results.

Let $\mathscr{X} = C[a,\chi]$ be the Banach space of continuous function $u:[a,\chi]\to\mathbb{R}$ with the norm $\|u\| = \max\{|u(\varrho)| : a \le \varrho \le \chi\}$. Clearly $\mathscr{X}$ is a Banach space with this norm and hence the product $\mathscr{X}\times\mathscr{X}$ is also a Banach space with the following norm

$$\|(u,v)\| = \|u\| + \|v\|.$$

Let $\gamma = \alpha + 2\beta - \alpha\beta$ such that $\alpha \in (1,2)$, $\beta \in [0,1]$ and let $\psi \in C^2[a,\chi]$ be an increasing function with $\psi'(\varrho) \neq 0$ for each $\varrho \in [a,\chi]$. We define the weighted space $\mathscr{X}_{2-\gamma;\psi}[a,\chi]$ of continuous function $u:[a,\chi]\to\mathbb{R}$ by

$$\mathscr{X}_{2-\gamma;\psi}[a,\chi]=\left\{u:(a,\chi]\to\mathbb{R}; [\psi(\varrho)-\psi(a)]^{2-\gamma}u(\varrho)\in\mathscr{X}\right\},\quad 1\le\gamma<2.$$

Obviously, $\mathscr{X}_{2-\gamma;\psi}[a,\chi]$ is a Banach space endowed with the norm

$$\|u\|_{2-\gamma;\psi} = \max_{\varrho\in[a,b]}\left|[\psi(\varrho)-\psi(a)]^{2-\gamma}u(\varrho)\right|.$$

Let $\mathscr{X}_{2-\gamma;\psi} := \mathscr{X}_{2-\gamma_1;\psi}\times\mathscr{X}_{2-\gamma_2;\psi}$, be the product weighted space with the norm

$$\|(u,v)\|_{2-\gamma;\psi} = \|u\|_{2-\gamma_1;\psi} + \|v\|_{2-\gamma_2;\psi},$$

for each $(u,v)\in\mathscr{X}_{2-\gamma;\psi}$.

For brevity, we set

$$\mathscr{N}_\psi^{\mu-1}(\varrho,s)=\psi'(s)\left(\psi(\varrho)-\psi(s)\right)^{\mu-1},$$
$$\Theta_\psi^{\mu-1}(\varrho,a)=\left(\psi(\varrho)-\psi(a)\right)^{\mu-1},$$
$$\frac{1}{\Gamma(\mu)}\int_a^\varrho \mathscr{N}_\psi^{\mu-1}(\varrho,s)\,ds = \frac{(\psi(\varrho)-\psi(a))^\mu}{\Gamma(\mu+1)}.$$

Definition 2.1

[2]

Let $\alpha > 0$, $f \in L_1[a,b]$. Then, the generalized RL fractional integral of a function f of order α with respect to $\psi$is defined by

$$I_{a^+}^{\alpha,\psi} f(\varrho) = \frac{1}{\Gamma(\alpha)} \int_a^\varrho \mathcal{N}_\psi^\alpha(\varrho, s) f(s)\, ds.$$

Definition 2.2

[12]

Let $n-1<\alpha<n\in\mathbb{N}$, and $f,\psi\in C^n[a,\chi]$. Then the generalized Hilfer fractional derivative of a function f of order α and type $0\le\beta\le1$ with respect to ψ is defined by

$$^{H}D_{a^+}^{\alpha,\beta,\psi} f(\varrho) = I^{\beta(n-\alpha);\psi} f_\psi^{[n]} I_{a^+}^{(1-\beta)(n-\alpha),\psi} f(\varrho)$$
$$= I^{\beta(n-\alpha);\psi} f_\psi^{[n]} I_{a^+}^{n-\gamma,\psi} f(\varrho)$$
$$= I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi} f(\varrho), \quad \gamma=\alpha+n\beta-\alpha\beta,$$

where

$$D_{a^+}^{\gamma;\psi} f(\varrho) = f_\psi^{[n]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(\varrho), \quad \text{and} \quad f_\psi^{[n]} = \left(\frac{1}{\psi'(\varrho)}\frac{d}{d\varrho}\right)^n.$$

Lemma 2.3

[2], [12]

Let $\alpha,\beta>0$ and $\delta>0$. Then

$$I_{a^+}^{\alpha,\psi} I_{a^+}^{\beta,\psi} f(\varrho) = I_{a^+}^{\alpha+\beta,\psi} f(\varrho),$$

$$I_{a^+}^{\alpha,\psi} \Theta_\psi^{\delta-1}(\varrho,a) = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \Theta_\psi^{\alpha+\delta-1}(\varrho,a),$$

and

$$^{H}D_{a^+}^{\alpha,\beta,\psi} \Theta_\psi^{\gamma-1}(\varrho,a) = 0, \quad \gamma=\alpha+n\beta-\alpha\beta.$$

Lemma 2.4

[12]

If $f\in C^n[a,b]$, $n-1<\alpha<n$, and $0\le\beta\le1$, then

$$I_{a^+}^{\alpha;\psi}\ {}^{H}D_{a^+}^{\alpha,\beta,\psi} f(\varrho) = f(\varrho) - \sum_{k=1}^n \frac{\Theta_\psi^{\gamma-k}(\varrho,a)}{\Gamma(\gamma-k+1)} f_\psi^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a),$$
$$^{H}D_{a^+}^{\alpha,\beta,\psi} I_{a^+}^{\alpha;\psi} f(\varrho) = f(\varrho).$$

Lemma 2.5

[44] (Leray–Schauder Alternative)

Let $G : \mathscr{X} \to \mathscr{X}$ be a completely continuous operator and let $\xi(G) = \{y \in \mathscr{X} : y = \delta G(y), \delta \in [0,1]\}$. Then either the set $\xi(G)$ is unbounded or G has at least one fixed point.

Theorem 2.6

[45] (Banach Fixed Point Theorem)

Let $\mathscr{X}$ be a Banach space, $K \subset \mathscr{X}$ be closed, and $G : K \to K$ be a strict contraction, i.e., $\|G(x) - G(y)\| \le L\|x - y\|$ for some $0 < L < 1$ and all $x, y \in K$. Then G has a fixed point in K.

Lemma 2.7

Let $i = 1, 2$, $\gamma_i = \alpha_i + 2\beta_i - \alpha_i\beta_i$ such that $\alpha_i \in (1,2)$, $\beta_i \in [0,1]$, and $h_i : (a, \chi] \to \mathbb{R}$ be a continuous functions. If $(u_1, u_2) \in \mathscr{X}_{2-\gamma;\psi}$ satisfies

$$\begin{cases} D_{a^+}^{\alpha_1,\beta_1,\psi} u_1(\varrho) + L_1 D_{a^+}^{\alpha_1-1,\beta_1,\psi} u_1(\varrho) = h_1(\varrho), \quad \varrho \in \mathscr{J} \\ D_{a^+}^{\alpha_2,\beta_2,\psi} u_2(\varrho) + L_2 D_{a^+}^{\alpha_2-1,\beta_1,\psi} u_2(\varrho) = h_2(\varrho), \quad \varrho \in \mathscr{J} \\ u_1(a) = 0, \qquad u_1(\chi) = Q_1 \\ u_2(a) = 0, \qquad u_2(\chi) = Q_2. \end{cases} \tag{2.1}$$

Then

$$\begin{aligned} u_i(\varrho) &= Q_i \frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)} \\ &+ L_i \left[\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)} \int_a^\chi \psi'(s) u_i(s) ds - \int_a^\varrho \psi'(s) u_i(s) ds \right] \\ &+ \frac{1}{\Gamma(\alpha_i)} \left[\int_a^\varrho \mathscr{N}_\psi^{\alpha_i-1}(\varrho,s) h_i(s) ds - \frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)} \int_a^\chi \mathscr{N}_\psi^{\alpha_i-1}(\chi,s) h_i(s) ds \right]. \end{aligned} \tag{2.2}$$

Proof

From the first equation of (2.1), we have

$$D_{a^+}^{\alpha_1,\beta_1,\psi} u_1(\varrho) + L_1 D_{a^+}^{\alpha_1-1,\beta_1,\psi} u_1(\varrho) = h_1(\varrho). \tag{2.3}$$

Applying the $I_{a^+}^{\alpha_1,\psi}$ to both sides of (2.3), we get

$$I_{a^+}^{\alpha_1,\psi} D_{a^+}^{\alpha_1,\beta_1,\psi} u_1(\varrho) + L_1 I_{a^+}^{\alpha_1,\psi} D_{a^+}^{\alpha_1-1,\beta_1,\psi} u_1(\varrho) = I_{a^+}^{\alpha_1,\psi} h_1(\varrho).$$

By Lemma2.4 and Definition2.2, one has

$$u_1(\varrho) = \frac{D_{a^+}^{\gamma_1-1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}\Theta_\psi^{\gamma_1-1}(\varrho,a) + \frac{I_{a^+}^{2-\gamma_1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1-1)}\Theta_\psi^{\gamma_1-2}(\varrho,a) - L_1 I_{a^+}^{1,\psi} I_{a^+}^{\alpha_1-1,\psi} D_{a^+}^{\alpha_1-1,\beta_1,\psi}u_1(\varrho) + I_{a^+}^{\alpha_1,\psi}h_1(\varrho). \tag{2.4}$$

Put $q \coloneqq \alpha_1 - 1$. Due to $1 < \alpha_1 < 2 \Longrightarrow 0 < q < 1, p = q + \beta_1(1-q)$. The Eq.(2.4) becomes

$$u_1(\varrho) = \frac{D_{a^+}^{\gamma_1-1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}\Theta_\psi^{\gamma_1-1}(\varrho,a) + \frac{I_{a^+}^{2-\gamma_1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1-1)}\Theta_\psi^{\gamma_1-2}(\varrho,a) - L_1 I_{a^+}^{1,\psi} I_{a^+}^{q,\psi} D_{a^+}^{q,\beta_1,\psi}u_1(\varrho) + I_{a^+}^{\alpha_1,\psi}h_1(\varrho)$$

By Lemma 2.4, we get

$$\begin{aligned} u_1(\varrho) &= \frac{D_{a^+}^{\gamma_1-1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}\Theta_\psi^{\gamma_1-1}(\varrho,a) + \frac{I_{a^+}^{2-\gamma_1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1-1)}\Theta_\psi^{\gamma_1-2}(\varrho,a) \\ &\quad - L_1 I_{a^+}^{1,\psi}\left[u_1(\varrho) - \frac{I_{a^+}^{1-p,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(p)}\Theta_\psi^{p-1}(\varrho,a)\right] + I_{a^+}^{\alpha_1,\psi}h_1(\varrho) \\ &= \frac{D_{a^+}^{\gamma_1-1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}\Theta_\psi^{\gamma_1-1}(\varrho,a) + \frac{I_{a^+}^{2-\gamma_1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1-1)}\Theta_\psi^{\gamma_1-2}(\varrho,a) \\ &\quad - L_1 I_{a^+}^{1,\psi}u_1(\varrho) + \frac{L_1 I_{a^+}^{1-p,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(p+1)}\Theta_\psi^{p}(\varrho,a) + I_{a^+}^{\alpha_1,\psi}h_1(\varrho) \end{aligned} \tag{2.5}$$

Since

$$p = q + \beta_1(1-q) = \alpha_1 - 1 + \beta_1(2-\alpha_1) = \gamma_1 - 1,$$

Eq.(2.5) reduces to

$$\begin{aligned} u_1(\varrho) &= \frac{D_{a^+}^{\gamma_1-1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}\Theta_\psi^{\gamma_1-1}(\varrho,a) + \frac{I_{a^+}^{2-\gamma_1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1-1)}\Theta_\psi^{\gamma_1-2}(\varrho,a) \\ &\quad - L_1 I_{a^+}^{1,\psi}u_1(\varrho) + \frac{L_1 I_{a^+}^{2-\gamma_1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}\Theta_\psi^{\gamma_1-1}(\varrho,a) + I_{a^+}^{\alpha_1,\psi}h_1(\varrho) \\ &= c_1\Theta_\psi^{\gamma_1-1}(\varrho,a) + c_2\left[(\gamma_1-1)\Theta_\psi^{\gamma_1-2}(\varrho,a) + L_1\Theta_\psi^{\gamma_1-1}(\varrho,a)\right] \\ &\quad - L_1\int_a^\varrho \psi'(s)u_1(s)\,ds + \frac{1}{\Gamma(\alpha_1)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_1-1}(\varrho,s)h_1(s)\,ds, \end{aligned} \tag{2.6}$$

where $c_1 = \frac{D_{a^+}^{\gamma_1-1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}$, $c_2 = \frac{I_{a^+}^{2-\gamma_1,\psi}u_1(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_1)}$ are arbitrary constants. By the same way, one can obtain

$$u_2(\varrho) = c_3\Theta_\psi^{\gamma_2-1}(\varrho,a) + c_4\left[(\gamma_2-1)\Theta_\psi^{\gamma_2-2}(\varrho,a) + L_2\Theta_\psi^{\gamma_2-1}(\varrho,a)\right] - L_2\int_a^\varrho \psi'(s)u_2(s)\,ds + \frac{1}{\Gamma(\alpha_2)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_2-1}(\varrho,s)h_2(s)\,ds, \tag{2.7}$$

where $c_3 = \frac{D_{a^+}^{\gamma_1-1,\psi}u_2(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_2)}$, and $c_4 = \frac{I_{a^+}^{2-\gamma_1,\psi}u_2(\varrho)\big|_{\varrho=a}}{\Gamma(\gamma_2)}$ are arbitrary constants. Now, Making use of the initial conditions $(u_1(a) = 0, u_2(a) = 0)$ along with (2.6), (2.7), we get

$c_2 = c_4 = 0,$

and hence (2.6), (2.7) reduce to

$$u_1(\varrho) = c_1 \Theta_{\psi}^{\gamma_1-1}(\varrho,a) - L_1 \int_a^{\varrho} \psi'(s)\, u_1(s)\, ds + \frac{1}{\Gamma(\alpha_1)} \int_a^{\varrho} \mathcal{N}_{\psi}^{\alpha_1-1}(\varrho,s)\, h_1(s)\, ds \tag{2.8}$$

and

$$u_2(\varrho) = c_3 \Theta_{\psi}^{\gamma_2-1}(\varrho,a) - L_2 \int_a^{\varrho} \psi'(s)\, u_2(s)\, ds + \frac{1}{\Gamma(\alpha_2)} \int_a^{\varrho} \mathcal{N}_{\psi}^{\alpha_2-1}(\varrho,s)\, h_2(s)\, ds \tag{2.9}$$

Next, by the boundary condition $(u_1(\chi) = Q_1, u_2(\chi) = Q_2)$, together with (2.8), (2.9), we get

$$c_1 = \frac{1}{\Theta_{\psi}^{\gamma_1-1}(\chi,a)} \left(Q_1 + L_1 \int_a^{\chi} \psi'(s)\, u_1(s)\, ds - \frac{1}{\Gamma(\alpha_1)} \int_a^{\chi} \mathcal{N}_{\psi}^{\alpha_1-1}(\chi,s)\, h_1(s)\, ds \right),$$

and

$$c_3 = \frac{1}{\Theta_{\psi}^{\gamma_2-1}(\chi,a)} \left(Q_2 + L_2 \int_a^{\chi} \psi'(s)\, u_2(s)\, ds - \frac{1}{\Gamma(\alpha_2)} \int_a^{\chi} \mathcal{N}_{\psi}^{\alpha_2-1}(\chi,s)\, h_2(s)\, ds \right).$$

Substituting c_1, c_2, c_3, and c_4 in (2.6), (2.7), we get (2.2). By direct calculation, we obtain proof of the converse.□

Consider the continuous operator $G : \mathscr{X}_{2-\gamma;\psi} \to \mathscr{X}_{2-\gamma;\psi}$ defined by

$$G(u_1,u_2)(\varrho) = (G_1(u_1,u_2)(\varrho), G_2(u_1,u_2)(\varrho)), \tag{2.10}$$

where

$$\begin{aligned} G_1(u_1,u_2)(\varrho) &= Q_1 \frac{\Theta_{\psi}^{\gamma_1-1}(\varrho,a)}{\Theta_{\psi}^{\gamma_1-1}(\chi,a)} + L_1 \left[\frac{\Theta_{\psi}^{\gamma_1-1}(\varrho,a)}{\Theta_{\psi}^{\gamma_1-1}(\chi,a)} \int_a^{\chi} \psi'(s)\, u_1(s)\, ds - \int_a^{\varrho} \psi'(s)\, u_1(s)\, ds \right] \\ &+ \frac{1}{\Gamma(\alpha_1)} \left[\int_a^{\varrho} \mathcal{N}_{\psi}^{\alpha_1-1}(\varrho,s)\, f_1(s,u_1(s),u_2(s))\, ds \right. \\ &- \left. \frac{\Theta_{\psi}^{\gamma_1-1}(\varrho,a)}{\Theta_{\psi}^{\gamma_1-1}(\chi,a)} \int_a^{\chi} \mathcal{N}_{\psi}^{\alpha_1-1}(\chi,s)\, f_1(s,u_1(s),u_2(s))\, ds \right], \end{aligned} \tag{2.11}$$

and

$$G_2(u_1,u_2)(\varrho) \tag{2.12}$$
$$= Q_2 \frac{\Theta_\psi^{\gamma_2-1}(\varrho,a)}{\Theta_\psi^{\gamma_2-1}(\chi,a)} + L_2\left[\frac{\Theta_\psi^{\gamma_2-1}(\varrho,a)}{\Theta_\psi^{\gamma_2-1}(\chi,a)}\int_a^\chi \psi'(s)\,u_2(s)\,ds - \int_a^\varrho \psi'(s)\,u_2(s)\,ds\right]$$
$$+ \frac{1}{\Gamma(\alpha_2)}\left[\int_a^\varrho \mathcal{N}_\psi^{\alpha_2-1}(\varrho,s)\,f_2(s,u_1(s),u_2(s))\,ds\right.$$
$$\left. - \frac{\Theta_\psi^{\gamma_2-1}(\varrho,a)}{\Theta_\psi^{\gamma_2-1}(\chi,a)}\int_a^\chi \mathcal{N}_\psi^{\alpha_2-1}(\chi,s)\,f_2(s,u_1(s),u_2(s))\,ds\right].$$

We noted that the fixed points of the operator G are solutions of system (1.2).

3. Existence of solution

In this section, we consider a general type coupled system of Hilfer sequential FDEs (1.2) involving the arbitrary function ψ. To demonstrate our main results, the following hypotheses must be satisfied.

(H_1) The functions $f_i(\varrho,u_1,u_2)$ $(i=1,2)$ are Carathĕodory on $\mathcal{J}\times\mathcal{X}_{2-\gamma;\psi}$.

(H_2) There exist measurable and bounded functions $\tau_i:\mathcal{J}\to(0,\infty)$ with $\sup_{\varrho\in\mathcal{J}}|\tau_i(\varrho)|=\tau_i^*$, for $i=1,2$ such that

$$|f_i(\varrho,u_1,u_2)| \le \frac{\tau_i(\varrho)\max\{u_1,u_2\}}{1+|u_1|+|u_2|},\ \text{for each } (u_1,u_2)\in\mathcal{X}_{2-\gamma;\psi},\varrho\in\mathcal{J}.$$

(H_3) There exist constant numbers $\mathcal{L}_i>0, i=1,2$ such that

$$|f_i(\varrho,u_1,u_2)-f_i(\varrho,\widehat{u}_1,\widehat{u}_2)|$$
$$\le \mathcal{L}_i(|u_1-\widehat{u}_1|+|u_2-\widehat{u}_2|),\ \text{for each } (u_1,u_2),(\widehat{u}_1,\widehat{u}_2)\in\mathcal{X}_{2-\gamma;\psi},\varrho\in\mathcal{J}.$$

In the following, we will applyLemma2.5 to give the existence result to system (1.2).

Theorem 3.1

Assume that (H_1)–(H_2) hold. If

$$\Lambda_1 := \max\{N_1,N_2\}<1,\quad \text{where}\quad N_i = 2\left(\frac{L_i\Theta_\psi^1(\chi,a)}{\gamma_i-1}\right),\ \text{for } i=1,2, \tag{3.1}$$

then the system (1.2) has at least one solution on $\mathcal{J}$.

Proof

We will prove that the operator G defined by (2.10) has a fixed point by using Lemma2.5. For that, we divide the proof into the following steps.

Step 1: G is a continuous.

Let (u_{1n},u_{2n}) be a sequence such that $(u_{1n},u_{2n})\to(u_1,u_2)$ in $\mathcal{X}_{2-\gamma;\psi}$. Then, for $\varrho\in\mathcal{J}$ and

$i = 1, 2$, we have

$$\left|[G_i(u_{1n}, u_{2n})(\varrho) - G_i(u_1, u_2)(\varrho)]\Theta_\psi^{2-\gamma_i}(\varrho, a)\right|$$
$$\leq L_i\left[\frac{\Theta_\psi^1(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi \psi'(s)\left|u_{in}(s) - u_i(s)\right| ds + \Theta_\psi^{2-\gamma_i}(\varrho, a)\int_a^\varrho \psi'(s)\left|u_{in}(s) - u_i(s)\right| ds\right]$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\varrho,a)}{\Gamma(\alpha_i)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_i-1}(\varrho, s)\left|f_i(s, u_{1n}(s), u_{2n}(s)) - f_i(s, u_1(s), u_2(s))\right| ds$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\chi,a)}{\Gamma(\alpha_i)}\int_a^\chi \mathcal{N}_\psi^{\alpha_i-1}(\chi, s)\left|f_i(s, u_{1n}(s), u_{2n}(s)) - f_i(s, u_1(s), u_2(s))\right| ds$$
$$\leq L_i\Big[\Theta_\psi^{2-\gamma_i}(\chi, a)\int_a^\chi \psi'(s)\,\Theta_\psi^{\gamma_i-2}(s, a)\left|\Theta_\psi^{2-\gamma_i}(s, a)(u_{in}(s) - u_i(s))\right| ds$$
$$+\Theta_\psi^{2-\gamma_i}(\chi, a)\int_a^\varrho \psi'(s)\,\Theta_\psi^{\gamma_i-2}(s, a)\left|\Theta_\psi^{2-\gamma_i}(s, a)(u_{in}(s) - u_i(s))\right| ds\Big]$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\varrho,a)}{\Gamma(\alpha_i)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_i-1}(\varrho, s)\left|f_i(s, u_{1n}(s), u_{2n}(s)) - f_i(s, u_1(s), u_2(s))\right| ds$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\chi,a)}{\Gamma(\alpha_i)}\int_a^\chi \mathcal{N}_\psi^{\alpha_i-1}(\chi, s)\left|f_i(s, u_{1n}(s), u_{2n}(s)) - f_i(s, u_1(s), u_2(s))\right| ds$$
$$\leq 2L_i\frac{\Theta_\psi^1(\chi,a)}{\gamma_i-1}\left\|u_{in} - u_i\right\|_{2-\gamma_i;\psi} + \frac{\Theta_\psi^{2-\gamma_i}(\varrho,a)}{\Gamma(\alpha_i)}$$
$$\int_a^\varrho \mathcal{N}_\psi^{\alpha_i-1}(\varrho, s)\left|f_i(s, u_{1n}(s), u_{2n}(s)) - f_i(s, u_1(s), u_2(s))\right| ds$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\chi,a)}{\Gamma(\alpha_i)}\int_a^\chi \mathcal{N}_\psi^{\alpha_i-1}(\chi, s)\left|f_i(s, u_{1n}(s), u_{2n}(s)) - f_i(s, u_1(s), u_2(s))\right| ds$$

Since $(u_{1n}, u_{2n})\to(u_1, u_2)$ as $n \to \infty$, and f_i are continuous, then by the Lebesgue dominated convergence theorem, we have

$$\left\|G_i(u_{1n}, u_{2n}) - G_i(u_1, u_2)\right\|_{2-\gamma_i;\psi} \to 0 \quad \text{as}(u_{1n}, u_{2n})\to(u_1, u_2). \tag{3.2}$$

It follows from (3.2) that

$$\left\|G(u_{1n}, u_{2n}) - G(u_1, u_2)\right\|_{2-\gamma;\psi}$$
$$= \left\|G_1(u_{1n}, u_{2n}) - G_1(u_1, u_2)\right\|_{2-\gamma_1;\psi} + \left\|G_2(u_{1n}, u_{2n}) - G_2(u_1, u_2)\right\|_{2-\gamma_2;\psi}$$
$$\to 0 \quad \text{as} \quad (u_{1n}, u_{2n})\to(u_1, u_2).$$

Hence G is continuous.

Step 2: G is compact.

Define a bounded, closed and convex set $\mathbb{B}_R = \left\{(u_1, u_2)\in \mathscr{X}_{2-\gamma;\psi} : \left\|(u_1, u_2)\right\|_{2-\gamma;\psi} \leq R\right\}$ with

$$R \geq \frac{\Lambda_2}{1-\Lambda_1} \quad \textit{where} \quad \Lambda_2 := \sum_{i=1}^2\left[Q_i\Theta_\psi^{2-\gamma_i}(\chi, a) + \frac{2\tau_i^*}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i+2-\gamma_i}(\chi, a)\right]. \tag{3.3}$$

First, we show that G is uniformly bounded on $\mathbb{B}_R$. For each $(u_1, u_2)\in \mathbb{B}_R$ and for $i = 1, 2$, we have

$$\left|G_i(u_1(\varrho), u_2(\varrho))\Theta_\psi^{2-\gamma_i}(\varrho, a)\right|$$
$$\leq Q_i \frac{\Theta_\psi^1(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}$$
$$+ L_i\left[\frac{\Theta_\psi^1(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi \psi'(s)\left|u_i(s)\right| ds + \Theta_\psi^{2-\gamma_i}(\varrho, a)\int_a^\varrho \psi'(s)\left|u_i(s)\right| ds\right]$$
$$+ \frac{\Theta_\psi^{2-\gamma_i}(\varrho,a)}{\Gamma(\alpha_i)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_i-1}(\varrho, s)\left|f_i(s, u_1(s), u_2(s))\right| ds$$
$$+ \frac{\Theta_\psi^{2-\gamma_i}(\chi,a)}{\Gamma(\alpha_i)}\int_a^\chi \mathcal{N}_\psi^{\alpha_i-1}(\chi, s)\left|f_i(s, u_1(s), u_2(s))\right| ds$$
$$\leq Q_i\Theta_\psi^{2-\gamma_i}(\chi, a)+L_i\left[\Theta_\psi^{2-\gamma_i}(\chi, a)\int_a^\chi \psi'(s)\,\Theta_\psi^{\gamma_i-2}(s, a)\left|\Theta_\psi^{2-\gamma_i}(s, a)u_i(s)\right| ds\right.$$
$$\left.+ \Theta_\psi^{2-\gamma_i}(\chi, a)\int_a^\varrho \psi'(s)\,\Theta_\psi^{\gamma_i-2}(s, a)\left|\Theta_\psi^{2-\gamma_i}(s, a)u_i(s)\right| ds\right]$$
$$+ \frac{\Theta_\psi^{2-\gamma_i}(\varrho,a)}{\Gamma(\alpha_i)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_i-1}(\varrho, s)\,\tau_i(s)\, ds + \frac{\Theta_\psi^{2-\gamma_i}(\chi,a)}{\Gamma(\alpha_i)}\int_a^\chi \mathcal{N}_\psi^{\alpha_i-1}(\chi, s)\,\tau_i(s)\, ds$$
$$\leq Q_i\Theta_\psi^{2-\gamma_i}(\chi, a)+2L_i\frac{\Theta_\psi^1(\chi,a)}{\gamma_i-1}\left\|u_i\right\|_{2-\gamma_i;\psi} + 2\frac{\tau_i^*}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i+2-\gamma_i}(\chi, a)$$
$$\leq Q_i\Theta_\psi^{2-\gamma_i}(\chi, a)+2\left[L_i\left\|u_i\right\|_{2-\gamma_i;\psi}\frac{\Theta_\psi^1(\chi,a)}{\gamma_i-1} + \frac{\tau_i^*}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i+2-\gamma_i}(\chi, a)\right]$$

Hence for $\varrho \in \mathscr{J}$ we have

$$\left\|G_i(u_1, u_2)\right\|_{2-\gamma_i;\psi} \leq Q_i\Theta_\psi^{2-\gamma_i}(\chi, a)$$
$$+2\left[L_i\left\|u_i\right\|_{2-\gamma_i;\psi}\frac{\Theta_\psi^1(\chi,a)}{\gamma_i-1} + \frac{\tau_i^*}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i+2-\gamma_i}(\chi, a)\right]. \tag{3.4}$$

From (3.4), then we have for each $(u_1, u_2)\in \mathbb{B}_R$,

$$\left\|G(u_1, u_2)\right\|_{2-\gamma;\psi} = \left\|G_1(u_1, u_2)\right\|_{2-\gamma_1;\psi} + \left\|G_2(u_1, u_2)\right\|_{2-\gamma_2;\psi}$$
$$\leq \sum_{i=1}^2\left[Q_i\Theta_\psi^{2-\gamma_i}(\chi, a)+\frac{2\tau_i^*}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i+2-\gamma_i}(\chi, a)\right]$$
$$+ 2\max\left\{\frac{L_1}{\gamma_1-1}, \frac{L_2}{\gamma_2-1}\right\}\left(\left\|u_1\right\|_{2-\gamma_1;\psi} + \left\|u_2\right\|_{2-\gamma_2;\psi}\right)\underset{\psi}{\overset{1}{\Theta}}(\chi, a)$$
$$\leq \sum_{i=1}^2\left[Q_i\Theta_\psi^{2-\gamma_i}(\chi, a)+\frac{2\tau_i^*}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i+2-\gamma_i}(\chi, a)\right]$$
$$+ 2\max\left\{\frac{L_1}{\gamma_1-1}, \frac{L_2}{\gamma_2-1}\right\}\left\|(u_1, u_2)\right\|_{2-\gamma;\psi}\Theta_\psi^1(\chi, a)$$
$$\leq \Lambda_2 + \Lambda_1 R \leq R.$$

Hence G is uniformly bounded. Now, we show that G is equicontinuous. Let $\varrho_1, \varrho_2 \in \mathscr{J}$ such that $\varrho_1 < \varrho_2$. Then, for $i = 1, 2$ we have

$$\left|G_i(u_1(\varrho_2), u_2(\varrho_2))\Theta_\psi^{2-\gamma_i}(\varrho_2, a) - G_i(u_1(\varrho_1), u_2(\varrho_1))\Theta_\psi^{2-\gamma_i}(\varrho_1, a)\right| \tag{3.5}$$

$$\le Q_i \frac{\Theta_\psi^1(\varrho_2,\varrho_1)}{\Theta_\psi^{\gamma_i-1}(\chi,a)} + L_i \frac{\Theta_\psi^1(\varrho_2,\varrho_1)}{\Theta_\psi^{\gamma_i-1}(\chi,a)} \int_a^\chi \psi'(s)\,\Theta_\psi^{\gamma_i-2}(s,a)\|u_i\|_{2-\gamma_i;\psi} ds$$

$$+ L_i\left(\Theta_\psi^{2-\gamma_i}(\varrho_2, a) - \Theta_\psi^{2-\gamma_i}(\varrho_1, a)\right)\int_a^{\varrho_1} \psi'(s)\,\Theta_\psi^{\gamma_i-2}(s,a)\|u_i\|_{2-\gamma_i;\psi} ds$$

$$+ L_i\Theta_\psi^{2-\gamma_i}(\varrho_2, a)\int_{\varrho_1}^{\varrho_2} \psi'(s)\,\Theta_\psi^{\gamma_i-2}(s,a)\|u_i\|_{2-\gamma_i;\psi} ds$$

$$+ \frac{1}{\Gamma(\alpha_i)}$$

$$\left[\int_a^{\varrho_1}\left(\mathscr{N}_\psi^{\alpha_i-1}(\varrho_2,s)\,\Theta_\psi^{2-\gamma_i}(\varrho_2,a) - \mathscr{N}_\psi^{\alpha_i-1}(\varrho_1,s)\,\Theta_\psi^{2-\gamma_i}(\varrho_1,a)\right)|f_i(s,u_1(s),u_2(s))|\,ds\right.$$

$$+ \Theta_\psi^{2-\gamma_i}(\varrho_2, a)\int_{\varrho_1}^{\varrho_2} \mathscr{N}_\psi^{\alpha_i-1}(\varrho_2,s)\,|f_i(s,u_1(s),u_2(s))|\,ds$$

$$\left.+ \frac{\Theta_\psi^1(\varrho_2,\varrho_1)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi \mathscr{N}_\psi^{\alpha_i-1}(\chi,s)\,|f_i(s,u_1(s),u_2(s))|\,ds\right]$$

$$\le Q_i \frac{\Theta_\psi^1(\varrho_2,\varrho_1)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}$$

$$+ \frac{L_i\|u_i\|_{2-\gamma_i;\psi}}{(\gamma_i-1)}\left[\Theta_\psi^1(\varrho_2,\varrho_1) + \left(\Theta_\psi^{2-\gamma_i}(\varrho_2,a) -_i \Theta_\psi^{2-\gamma_i}(\varrho_1,a)\right)\Theta_\psi^{\gamma_i-1}(\varrho_2,\varrho_1)\right]$$

$$+ \frac{L_i\|u_i\|_{2-\gamma_i;\psi}}{(\gamma_i-1)}\left[\Theta_\psi^{2-\gamma_i}(\varrho_2,a)\Theta_\psi^{\gamma_i-1}(\varrho_2,\varrho_1)\right]$$

$$+ \frac{\tau_i^*}{\Gamma(\alpha_i+1)}\left[\Theta_\psi^{2+\alpha_i-\gamma_i}(\varrho_2,a) - \Theta_\psi^{2+\alpha_i-\gamma_i}(\varrho_1,a) - \Theta_\psi^{2-\gamma_i}(\varrho_2,a)\Theta_\psi^{\alpha_i}(\varrho_2,\varrho_1)\right.$$

$$\left.+ \Theta_\psi^{2-\gamma_i}(\varrho_2,a)\Theta_\psi^{\alpha_i}(\varrho_2,\varrho_1) + \Theta_\psi^1(\varrho_2,\varrho_1)\Theta_\psi^{\alpha_i-\gamma_i+1}(\chi,a)\right]$$

Take $\varrho_2 \to \varrho_1$, from (3.5), we have

$$\|G_i(u_1(\varrho_2), u_2(\varrho_2)) - G_i(u_1(\varrho_1), u_2(\varrho_1))\|_{2-\gamma_i;\psi} \to 0 \text{ as } \varrho_2 \to \varrho_1. \tag{3.6}$$

It follows from (3.6) that

$$\|G(u_1(\varrho_2), u_2(\varrho_2)) - G(u_1(\varrho_1), u_2(\varrho_1))\|_{2-\gamma;\psi} \to 0 \text{ as } \varrho_2 \to \varrho_1.$$

Hence G is equicontinuous. By Arzelá–Ascoli theorem, we infer that G is compact in $\mathscr{X}_{2-\gamma;\psi}$. Therefore, from the above steps, we conclude that G is completely continuous.

Step 3: The set $\delta = \{(u_1, u_2) \in \mathscr{X}_{2-\gamma;\psi} : (u_1, u_2) = \xi G(u_1, u_2), \quad \xi \in (0,1)\}$ is bounded.

Let $(u_1, u_2) \in \delta$. Then $(u_1, u_2) = \xi G(u_1, u_2)$. Now, for $\varrho \in \mathscr{J}$, we have $u_1(\varrho) = \xi G_1(u_1, u_2)$ and $u_2(\varrho) = \xi G_2(u_1, u_2)$. According to our hypotheses and for $i = 1, 2$, we attain

$$\left|u_i(\varrho)\Theta_\psi^{2-\gamma_i}(\varrho,a)\right| = \left|\xi\Theta_\psi^{2-\gamma_i}(\varrho,a)G_i(u_1,u_2)\right|$$
$$\leq Q_i\Theta_\psi^{2-\gamma_i}(\chi,a)+\frac{2L_i\|u_i\|_{2-\gamma_i;\psi}}{\gamma_i-1}\Theta_\psi^1(\chi,a)$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\varrho,a)}{\Gamma(\alpha_i)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_i-1}(\varrho,s)\left|f_i(s,u_1(s),u_2(s))\right|ds$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\chi,a)}{\Gamma(\alpha_i)}\int_a^\chi \mathcal{N}_\psi^{\alpha_i-1}(\chi,s)\left|f_i(s,u_1(s),u_2(s))\right|ds$$
$$\leq Q_i\Theta_\psi^{2-\gamma_i}(\chi,a)+\frac{2L_i\|u_i\|_{2-\gamma_i;\psi}}{\gamma_i-1}\Theta_\psi^1(\chi,a)$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\varrho,a)}{\Gamma(\alpha_i)}\int_a^\varrho \mathcal{N}_\psi^{\alpha_i-1}(\varrho,s)\tau_i(s)ds$$
$$+\frac{\Theta_\psi^{2-\gamma_i}(\chi,a)}{\Gamma(\alpha_i)}\int_a^\chi \mathcal{N}_\psi^{\alpha_i-1}(\chi,s)\tau_i(s)ds$$
$$\leq Q_i\Theta_\psi^{2-\gamma_i}(\chi,a)+\frac{2L_i\|u_i\|_{2-\gamma_i;\psi}}{\gamma_i-1}\Theta_\psi^1(\chi,a)$$
$$+\frac{2\tau_i^*\Theta_\psi^{2-\gamma_i+\alpha_i}(\chi,a)}{\Gamma(\alpha_i+1)},$$

which, on taking maximum for $\varrho\in\mathcal{J}$, yields

$$\|u_i\|_{2-\gamma_i;\psi}\leq Q_i\Theta_\psi^{2-\gamma_i}(\chi,a)+\frac{2L_i\|u_i\|_{2-\gamma_i;\psi}}{\gamma_i-1}\Theta_\psi^1(\chi,a)$$
$$+\frac{2\tau_i^*\Theta_\psi^{2-\gamma_i+\alpha_i}(\chi,a)}{\Gamma(\alpha_i+1)}. \tag{3.7}$$

From (3.7), we have

$$\|(u_1,u_2)\|_{2-\gamma;\psi}=\|u_1\|_{2-\gamma_1;\psi}+\|u_2\|_{2-\gamma_2;\psi}$$
$$\leq\sum_{i=1}^2\left[Q_i\Theta_\psi^{2-\gamma_i}(\chi,a)+\frac{2\tau_i^*\Theta_\psi^{2-\gamma_i+\alpha_i}(\chi,a)}{\Gamma(\alpha_i+1)}\right]$$
$$+2\left(\frac{L_1\Theta_\psi^1(\chi,a)}{\gamma_1-1}\right)\|u_1\|_{2-\gamma_1,\psi}+2\left(\frac{L_2\Theta_\psi^1(\chi,a)}{\gamma_2-1}\right)\|u_2\|_{2-\gamma_2;\psi}$$
$$\leq\sum_{i=1}^2\left[Q_i\Theta_\psi^{2-\gamma_i}(\chi,a)+\frac{2\tau_i^*\Theta_\psi^{2-\gamma_i+\alpha_i}(\chi,a)}{\Gamma(\alpha_i+1)}\right]$$
$$+\max\{N_1,N_2\}\|(u_1,u_2)\|_{2-\gamma;\psi}$$
$$=\Lambda_2+\Lambda_1\|(u_1,u_2)\|_{2-\gamma;\psi}.$$

Since $\Lambda_1<1$, we get

$$\|(u_1,u_2)\|_{2-\gamma;\psi}\leq\frac{\Lambda_2}{1-\Lambda_1}\leq R.$$

Hence, the set δ is bounded. According to the above steps together with Lemma2.5, we conclude that G has at least one fixed point. Consequently, the system (1.2) has at least one solution on $\mathcal{J}$.□

In the following theorem, we prove the uniqueness of solutions to system (1.2) by using Theorem2.6.

Theorem 3.2

Assume that (H_1)*–*(H_3) *hold. If* $\sigma = \max\{\sigma_1, \sigma_2\} < 1$*, then, the system* (1.2) *has a* *unique solution, where*

$$\sigma_1 = 2\left(\frac{L_1\Theta_\psi^1(\chi,a)}{\gamma_1-1} + \frac{\mathscr{L}_1\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_1-1)} + \frac{\mathscr{L}_2\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_2+\gamma_1-\gamma_2}(\chi,a)}{\Gamma(\alpha_2+\gamma_1-1)}\right),$$

$$\sigma_2 = 2\left(\frac{L_2\Theta_\psi^1(\chi,a)}{\gamma_2-1} + \frac{\mathscr{L}_2\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_2}(\chi,a)}{\Gamma(\alpha_2+\gamma_2-1)} + \frac{\mathscr{L}_1\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_1+\gamma_2-\gamma_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_2-1)}\right).$$

Proof

Define a bounded, closed and convex set $\Bbbk_R = \left\{(u_1,u_2)\in \mathscr{X}_{2-\gamma;\psi} : \|(u_1,u_2)\|_{2-\gamma,\psi} \le R\right\}$ where R defined as in Theorem3.1. First, we show that $G(\Bbbk_R) \subset \Bbbk_R$. By the second step in Theorem3.1, we have $G(\Bbbk_R) \subset \Bbbk_R$. Next, we need to prove that G is contraction map. Indeed, for $(u_1,u_2),(x_1,x_2) \in \mathscr{X}_{2-\gamma;\psi}$ and $\varrho \in \mathscr{J}$, we obtain

$$\begin{aligned}
&\left|(G_1(u_1(\varrho),u_2(\varrho)) - G_1(x_1(\varrho),x_2(\varrho)))\Theta_\psi^{2-\gamma_1}(\varrho,a)\right| \\
&\le L_1\left[\Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\chi \psi'(s)\left|u_1(s)-x_1(s)\right|ds + \Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\varrho \psi'(s)\left|u_1(s)-x_1(s)\right|ds\right] \\
&+ \frac{\Theta_\psi^{2-\gamma_1}(\varrho,a)}{\Gamma(\alpha_1)}\left[\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho,s)\left|f_1(s,u_1(s),u_2(s)) - f(s,x_1(s),x_2(s))\right|ds\right. \\
&\left.+ \Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\chi \mathscr{N}_\psi^{\alpha_1-1}(\chi,s)\left|f_1(s,u_1(s),u_2(s)) - f_1(s,x_1(s),x_2(s))\right|ds\right] \\
&\le L_1\left[\Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\chi \psi'(s)\left|u_1(s)-x_1(s)\right|ds + \Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\varrho \psi'(s)\left|u_1(s)-x_1(s)\right|ds\right] \\
&+ \frac{\mathscr{L}_1\Theta_\psi^{2-\gamma_1}(\varrho,a)}{\Gamma(\alpha_1)}\left[\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho,s)(\left|u_1(s)-x_1(s)\right| + \left|u_2(s)-x_2(s)\right|)ds\right. \\
&\left.+ \Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\chi \mathscr{N}_\psi^{\alpha_1-1}(\chi,s)(\left|u_1(s)-x_1(s)\right| + \left|u_2(s)-x_2(s)\right|)ds\right] \\
&\le \frac{2L_1\|u_1-x_1\|_{2-\gamma_1,\psi}\Theta_\psi^1(\chi,a)}{\gamma_1-1} \\
&+ \\
&\frac{\mathscr{L}_1\Theta_\psi^{2-\gamma_1}(\varrho,a)}{\Gamma(\alpha_1)} \\
&\left[\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho,s)\left|u_1(s)-x_1(s)\right|ds + \int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho,s)\left|u_2(s)-x_2(s)\right|ds\right. \\
&+ \Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\chi \mathscr{N}_\psi^{\alpha_1-1}(\chi,s)\left|u_1(s)-x_1(s)\right| \\
&\left.+ \Theta_\psi^{2-\gamma_1}(\chi,a)\int_a^\chi \mathscr{N}_\psi^{\alpha_1-1}(\chi,s)\left|u_2(s)-x_2(s)\right|ds\right] \\
&\le \frac{2L_1\|u_1-x_1\|_{2-\gamma_1,\psi}\Theta_\psi^1(\chi,a)}{\gamma_1-1} + \frac{2\mathscr{L}_1\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_1-1)}\|u_1-x_1\|_{2-\gamma_1,\psi} \\
&+ \frac{2\mathscr{L}_1\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_1+\gamma_2-\gamma_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_2-1)}\|u_2-x_2\|_{2-\gamma_2,\psi}
\end{aligned}$$

and consequently we obtain

$$\|G_1(u_1,u_2)-G_1(x_1,x_2)\|_{2-\gamma_1,\psi} \tag{3.8}$$

$$\leq \frac{2L_1\|u_1-x_1\|_{2-\gamma_1,\psi}\Theta_\psi^1(\chi,a)}{\gamma_1-1}+\frac{2\mathscr{L}_1\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_1-1)}\|u_1-x_1\|_{2-\gamma_1,\psi} \tag{3.9}$$
$$+\frac{2\mathscr{L}_1\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_1+\gamma_2-\gamma_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_2-1)}\|u_2-x_2\|_{2-\gamma_2,\psi}.$$

By the same way, one can obtain

$$\begin{aligned}&\|G_2(u_1,u_2)-G_2(x_1,x_2)\|_{2-\gamma_2,\psi}\\ &\leq \frac{2L_2\|u_2-x_2\|_{2-\gamma_2,\psi}\Theta_\psi^1(\chi,a)}{\gamma_2-1}+\frac{2\mathscr{L}_2\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_2}(\chi,a)}{\Gamma(\alpha_2+\gamma_2-1)}\|u_2-x_2\|_{2-\gamma_2,\psi}\\ &+\frac{2\mathscr{L}_2\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_2+\gamma_1-\gamma_2}(\chi,a)}{\Gamma(\alpha_2+\gamma_1-1)}\|u_1-x_1\|_{2-\gamma_1,\psi}.\end{aligned} \tag{3.10}$$

It follows from (3.9), (3.10) that

$$\begin{aligned}&\|G(u_1,u_2)-G(x_1,x_2)\|_{2-\gamma,\psi}\\ &=\|G_1(u_1,u_2)-G_1(x_1,x_2)\|_{2-\gamma_1,\psi}+\|G_2(u_1,u_2)-G_2(x_1,x_2)\|_{2-\gamma_2,\psi}\\ &\leq 2\left(\frac{L_1\Theta_\psi^1(\chi,a)}{\gamma_1-1}+\frac{\mathscr{L}_1\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_1-1)}+\frac{\mathscr{L}_2\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_2+\gamma_1-\gamma_2}(\chi,a)}{\Gamma(\alpha_2+\gamma_1-1)}\right)\|u_1-x_1\|_{2-\gamma_1,\psi}\\ &+2\left(\frac{L_2\Theta_\psi^1(\chi,a)}{\gamma_2-1}+\frac{\mathscr{L}_2\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_2}(\chi,a)}{\Gamma(\alpha_2+\gamma_2-1)}+\frac{\mathscr{L}_1\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_1+\gamma_2-\gamma_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_2-1)}\right)\|u_2-x_2\|_{2-\gamma_2,\psi}\\ &\leq\sigma\left(\|u_1-x_1\|_{2-\gamma_1,\psi}+\|u_2-x_2\|_{2-\gamma_2,\psi}\right).\end{aligned}$$

Thus, the operator G is a contraction. So, by Theorem2.6, the system (1.2) has a unique solution.□

4. Ulam–Hyers stability

In this section, we discuss the Ulam–Hyers stability of the system ((1.2)). The following observations are taken from[17].

Remark 4.1

A functions $(\widehat{u_1},\widehat{u_2})\in\mathscr{X}_{2-\gamma;\psi}$ satisfies the inequalities

$$\begin{aligned}&\left|D_{a^+}^{\alpha_i,\beta_i,\psi}\widehat{u_i}(\varrho)+L_iD_{a^+}^{\alpha_i-1,\beta_i,\psi}\widehat{u_i}(\varrho)-f_i(\varrho,\widehat{u_1}(\varrho),\widehat{u_2}(\varrho))\right|\leq\varepsilon_i,\\ &\varrho\in\mathscr{J},\quad\text{for } i=1,2.\end{aligned} \tag{4.1}$$

if and only if there exists a functions $\eta_i\in\mathscr{X}_{2-\gamma,\psi}$, $i=1,2$ such that

(i)

$$|\eta_i(\varrho)|\leq\varepsilon_i,\qquad\varrho\in\mathscr{J}\qquad;$$

(ii)

$$D_{a^+}^{\alpha_i,\beta_i,\psi}\widehat{u_i}(\varrho)+L_iD_{a^+}^{\alpha_i-1,\beta_i,\psi}\widehat{u_i}(\varrho)=f_i(\varrho,\widehat{u_1}(\varrho),\widehat{u_2}(\varrho))+\eta_i(\varrho),.\\ \varrho\in\mathscr{J}$$

Definition 4.2

The system (1.2) is Ulam–Hyers stable if there exists $K>0$ such that, for each $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}>0$ and each $(\widehat{u_1},\widehat{u_2})\in\mathscr{X}_{2-\gamma;\psi}$ satisfies the inequalities (4.1), there exists a solution $(u_1,u_2)\in\mathscr{X}_{2-\gamma,\psi}$ of the system (1.2) with

$$\|(\widehat{u_1},\widehat{u_2})-(u_1,u_2)\|_{1-\gamma,\psi}\le K\varepsilon,\qquad\varrho\in\mathscr{J}.$$

Lemma 4.3

Let $\alpha_i\in(1,2)$, $\beta_i\in[0,1]$, $i=1,2$. If a function $(\widehat{u_1},\widehat{u_2})\in\mathscr{X}_{2-\gamma;\psi}$ satisfies the inequalities (4.1), then $(\widehat{u_1},\widehat{u_2})$ satisfies the following integral inequalities

$$\left|\widehat{u_i}(\varrho)-\mathscr{A}_{\widehat{u_i}}-\frac{1}{\Gamma(\alpha_i)}\left[\int_a^\varrho\mathscr{N}_\psi^{\alpha_i-1}(\varrho,s)f_i(s,\widehat{u_1}(s),\widehat{u_2}(s))ds\right]\right|\\ \le\frac{\varepsilon_i}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i}(\chi,a)\left[1-\Theta_\psi^{2-\gamma_i}(\chi,a)\right]$$

where

$$\mathscr{A}_{\widehat{u_i}}:=Q_i\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}+L_i\left[\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi\psi'(s)\widehat{u_i}(s)ds-\int_a^\varrho\psi'(s)\widehat{u_i}(s)ds\right]\\ -\frac{1}{\Gamma(\alpha_i)}\left[\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi\mathscr{N}_\psi^{\alpha_i-1}(\chi,s)f_i(s,\widehat{u_1}(s),\widehat{u_2}(s))ds\right].$$

Proof

Indeed by Remark4.1, we have

$$D_{a^+}^{\alpha_i,\beta_i,\psi}\widehat{u_i}(\varrho)+L_iD_{a^+}^{\alpha_i-1,\beta_i,\psi}\widehat{u_i}(\varrho)=f_i(\varrho,\widehat{u_1}(\varrho),\widehat{u_2}(\varrho))+\eta_i(\varrho),\quad \varrho\in\mathscr{J}.$$

Then

$$\widehat{u_i}(\varrho)=Q_1\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}+L_i\left[\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi\psi'(s)u_i(s)ds-\int_a^\varrho\psi'(s)u_i(s)ds\right]+\frac{1}{\Gamma(\alpha_i)}\left[\int_a^\varrho\mathscr{N}_\psi^{\alpha_i-1}(\varrho,s)f_i(s,\widehat{u_1}(s),\widehat{u_2}(s))ds-\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi\mathscr{N}_\psi^{\alpha_i-1}(\chi,s)f_i(s,\widehat{u_1}(s),\widehat{u_2}(s))ds+\int_a^\varrho\mathscr{N}_\psi^{\alpha_i-1}(\varrho,s)\eta_i(s)ds-\frac{\Theta_\psi^{\gamma_i-1}(\varrho,a)}{\Theta_\psi^{\gamma_i-1}(\chi,a)}\int_a^\chi\mathscr{N}_\psi^{\alpha_i-1}(\chi,s)\eta_i(s)ds\right].$$

It follows that

$$\begin{aligned}&\left|\widehat{u_i}(\varrho)-\mathscr{A}_{\widehat{u_i}}-\frac{1}{\Gamma(\alpha_i)}\left[\int_a^\varrho\mathscr{N}_\psi^{\alpha_i-1}(\varrho,s)f_i(s,\widehat{u_1}(s),\widehat{u_2}(s))ds\right]\right|\\ &\le\frac{1}{\Gamma(\alpha_i)}\left[\int_a^\varrho\mathscr{N}_\psi^{\alpha_i-1}(\varrho,s)|\eta_i(s)|ds-\Theta_\psi^{2-\gamma_i}(\chi,a)\int_a^\chi\mathscr{N}_\psi^{\alpha_i-1}(\chi,s)|\eta_i(s)|ds\right]\\ &\le\frac{\varepsilon_i}{\Gamma(\alpha_i)}\left[\int_a^\varrho\mathscr{N}_\psi^{\alpha_i-1}(\varrho,s)ds-\Theta_\psi^{2-\gamma_i}(\chi,a)\int_a^\chi\mathscr{N}_\psi^{\alpha_i-1}(\chi,s)ds\right]\\ &=\frac{\varepsilon_i}{\Gamma(\alpha_i+1)}\left[\Theta_\psi^{\alpha_i}(\varrho,a)-\Theta_\psi^{2-\gamma_i+\alpha_i}(\chi,a)\right]\\ &\le\frac{\varepsilon_i}{\Gamma(\alpha_i+1)}\Theta_\psi^{\alpha_i}(\chi,a)\left[1-\Theta_\psi^{2-\gamma_i}(\chi,a)\right].\quad\square\end{aligned}$$

In the forthcoming theorem, we prove the stability results for the system (1.2).

Theorem 4.4

Assume that (H_1) *and* (H_3) *hold. Then*

$$D_{a^+}^{\alpha_i,\beta_i,\psi}u_i(\varrho)+L_iD_{a^+}^{\alpha_i-1,\beta_i,\psi}u_i(\varrho)=f_i(\varrho,u_1(\varrho),u_2(\varrho)),\qquad\varrho\in\mathscr{J},\tag{4.2}$$

is Ulam–Hyers stable, provided that $\triangle=-M_2(1-S_1)+M_1(1-S_2)\neq0$, *where*

$$S_1=\frac{\mathscr{L}_1\Theta_\psi^{\alpha_1}(\chi,a)}{\Gamma(\alpha_1+1)},\quad S_2=\frac{\mathscr{L}_2\Theta_\psi^{\alpha_2}(\chi,a)}{\Gamma(\alpha_2+1)}$$

$$M_1=\frac{\mathscr{L}_1\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_1+\gamma_2-\gamma_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_2-1)},$$

$$M_2=\frac{\mathscr{L}_2\Gamma(\gamma_1-1)\Theta_\psi^{\alpha_2+\gamma_1-\gamma_2}(\chi,a)}{\Gamma(\alpha_2+\gamma_1-1)}.$$

Proof

Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and $(\widehat{u_1}, \widehat{u_2}) \in \mathscr{X}_{2-\gamma;\psi}$ be a functions satisfying the inequalities

$$\left|D_{a^+}^{\alpha_i,\beta_i,\psi}\widehat{u_i}(\varrho) + L_i D_{a^+}^{\alpha_i-1,\beta_i,\psi}\widehat{u_i}(\varrho) - f_i(\varrho, \widehat{u_1}(\varrho), \widehat{u_2}(\varrho))\right| \le \varepsilon_i, \quad \varrho \in \mathscr{J}, i = 1, 2, \tag{4.3}$$

and let $(u_1, u_2) \in \mathscr{X}_{2-\gamma,\psi}$ be the unique solution of the following system

$$\begin{cases} D_{a^+}^{\alpha_1,\beta_1,\psi}u_1(\varrho) + L_1 D_{a^+}^{\alpha_1-1,\beta_1,\psi}u_1(\varrho) = f_1(\varrho, u_1(\varrho), u_2(\varrho)), \quad 1 < \alpha_1 < 2, \\ \quad \varrho \in \mathscr{J}, \\ D_{a^+}^{\alpha_2,\beta_2,\psi}u_2(\varrho) + L_2 D_{a^+}^{\alpha_2-1,\beta_1,\psi}u_2(\varrho) = f_2(\varrho, u_1(\varrho), u_2(\varrho)), \quad 1 < \alpha_2 < 2, \\ \quad \varrho \in \mathscr{J}, \\ \quad u_1(a) = \widehat{u_1}(a) = 0, \qquad u_1(\chi) = \widehat{u_1}(\chi) = Q_1, \\ \quad u_2(a) = \widehat{u_2}(a) = 0, \qquad u_2(\chi) = \widehat{u_2}(\chi) = Q_2. \end{cases}$$

Now, by using Theorem 3.1, we have

$$u_i(\varrho) = \mathscr{A}_{u_i} + \frac{1}{\Gamma(\alpha_i)}\left[\int_a^\varrho \mathscr{N}_\psi^{\alpha_i-1}(\varrho, s) f_i(s, u_1(s), u_2(s))\, ds\right], i = 1, 2.$$

Since

$$\begin{matrix} u_1(a) = \widehat{u_1}(a) = 0, & u_1(\chi) = \widehat{u_1}(\chi) = Q_1 \\ u_2(a) = \widehat{u_2}(a) = 0, & u_2(\chi) = \widehat{u_2}(\chi) = Q_2 \end{matrix},$$

we can easily prove that $\mathscr{A}_{u_1} = \mathscr{A}_{\widehat{u_1}}$ and $\mathscr{A}_{u_2} = \mathscr{A}_{\widehat{u_2}}$. Hence, from ($H_2$) and Lemma 4.3, then for each $\varrho \in (0, \chi]$, we have

$$\begin{aligned} &|\widehat{u_1}(\varrho) - u_1(\varrho)| \\ &\le \left|\widehat{u_1}(\varrho) - \mathscr{A}_{\widehat{u_1}} - \frac{1}{\Gamma(\alpha_1)}\left[\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho, s) f_1(s, \widehat{u_1}(s), \widehat{u_2}(s))\, ds\right]\right| \\ &+ \\ &\left|\frac{1}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho, s) f_1(s, \widehat{u_1}(s), \widehat{u_2}(s))\, ds \right. \\ &\left. - \frac{1}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho, s) f_1(s, u_1(s), u_2(s))\, ds\right| \\ &\le \varepsilon_1 \Theta_\psi^{\alpha_1}(\chi, a)\left[1 - \Theta_\psi^{2-\gamma_1}(\chi, a)\right] \\ &+ \frac{1}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho, s)\left|f_1(s, \widehat{u_1}(s), \widehat{u_2}(s)) - f_1(s, u_1(s), u_2(s))\right| ds \\ &\le \varepsilon_1 \Theta_\psi^{\alpha_1}(\chi, a)\left[1 - \Theta_\psi^{2-\gamma_1}(\chi, a)\right] \\ &+ \frac{\mathscr{L}_1}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho, s)(|\widehat{u_1}(s) - u_1(s)| + |\widehat{u_2}(s) - u_2(s)|)ds \\ &\le \varepsilon_1 \Theta_\psi^{\alpha_1}(\chi, a)\left[1 - \Theta_\psi^{2-\gamma_1}(\chi, a)\right] \\ &+ \frac{\mathscr{L}_1}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho, s)|\widehat{u_1}(s) - u_1(s)|\, ds \\ &+ \frac{\mathscr{L}_1}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho, s)|\widehat{u_2}(s) - u_2(s)|\, ds. \end{aligned} \tag{4.4}$$

Thus

$$\begin{aligned}
&\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}\\
&\leq \varepsilon_1 \Theta_\psi^{2-\gamma_1}(\varrho,a)\Theta_\psi^{\alpha_1}(\chi,a)\left[1-\Theta_\psi^{2-\gamma_1}(\chi,a)\right]\\
&+\frac{\mathscr{L}_1\Theta_\psi^{2-\gamma_1}(\varrho,a)}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho,s)\left|\left(\widehat{u_1}(s)-u_1(s)\right)\right|ds\\
&+\frac{\mathscr{L}_1\Theta_\psi^{2-\gamma_1}(\varrho,a)}{\Gamma(\alpha_1)}\int_a^\varrho \mathscr{N}_\psi^{\alpha_1-1}(\varrho,s)\left|\widehat{u_2}(s)-u_2(s)\right|ds\\
&\leq \varepsilon_1 \Theta_\psi^{2-\gamma_1+\alpha_1}(\chi,a)\left[1-\Theta_\psi^{2-\gamma_1}(\chi,a)\right]\\
&+\frac{\mathscr{L}_1\Theta_\psi^{\alpha_1}(\chi,a)}{\Gamma(\alpha_1+1)}\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}+\frac{\mathscr{L}_1\Gamma(\gamma_2-1)\Theta_\psi^{\alpha_1+\gamma_2-\gamma_1}(\chi,a)}{\Gamma(\alpha_1+\gamma_2-1)}\|\widehat{u_2}-u_2\|_{2-\gamma_2,\psi}.
\end{aligned}$$

Hence

$$(1-S_1)\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}-M_1\|\widehat{u_2}-u_2\|_{2-\gamma_2,\psi}\leq \varepsilon_1 K_1. \tag{4.5}$$

In the same technique, we get

$$(1-S_2)\|\widehat{u_2}-u_2\|_{2-\gamma_2,\psi}-M_2\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}\leq \varepsilon_2 K_2, \tag{4.6}$$

where

$$\begin{aligned}
K_1&=\Theta_\psi^{2-\gamma_1+\alpha_1}(\chi,a)\left[1-\Theta_\psi^{2-\gamma_1}(\chi,a)\right],\\
K_2&=\Theta_\psi^{2-\gamma_2+\alpha_1}(\chi,a)\left[1-\Theta_\psi^{2-\gamma_2}(\chi,a)\right].
\end{aligned}$$

Inequalities (4.5), (4.6) can be writing asmatrices as follows

$$\begin{pmatrix}(1-S_1) & -M_1\\ -M_2 & (1-S_2)\end{pmatrix}\begin{pmatrix}\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}\\ \|\widehat{u_2}-u_2\|_{2-\gamma_2,\psi}\end{pmatrix}\leq\begin{pmatrix}\varepsilon_1K_1\\ \varepsilon_2K_2\end{pmatrix}.$$

By simple computations, the above inequality becomes

$$\begin{pmatrix}\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}\\ \|\widehat{u_2}-u_2\|_{2-\gamma_2,\psi}\end{pmatrix}\leq\begin{pmatrix}\frac{(1-S_2)}{\triangle} & \frac{M_1}{\triangle}\\ \frac{M_2}{\triangle} & \frac{(1-S_1)}{\triangle}\end{pmatrix}\times\begin{pmatrix}\varepsilon_1K_1\\ \varepsilon_2K_2\end{pmatrix},$$

Since $\triangle\neq 0$. This leads to

$$\begin{aligned}
\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}&\leq\frac{(1-S_2)K_1}{\triangle}\varepsilon_1+\frac{M_1K_2}{\triangle}\varepsilon_2\\
\|\widehat{u_2}-u_2\|_{2-\gamma_2,\psi}&\leq\frac{M_2K_1}{\triangle}\varepsilon_1+\frac{(1-S_1)K_2}{\triangle}\varepsilon_2.
\end{aligned}$$

Thus

$$\begin{aligned}&\|(\widehat{u_1},\widehat{u_2})-(u_1,u_2)\|_{1-\gamma,\psi}\\&\leq\|\widehat{u_1}-u_1\|_{2-\gamma_1,\psi}+\|\widehat{u_2}-u_2\|_{2-\gamma_2,\psi}\\&\leq\left(\frac{(1-S_2)K_1+M_2K_1}{\triangle}\right)\varepsilon_1+\left(\frac{M_1K_2+(1-S_1)K_2}{\triangle}\right)\varepsilon_2\\&\leq\varepsilon K,\end{aligned}\tag{4.7}$$

where $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$ and

$$K=\left(\frac{M_2K_1+(1-S_2)K_1+M_1K_2+(1-S_1)K_2}{\triangle}\right).$$

Hence from the inequality (4.7) and Definition4.2 the solution of the coupled system (1.2) is Ulam–Hyers stable.□

5. An example

Consider the following coupled system of ψ-Hilfer sequential FDEs

$$\begin{cases}D_{a^+}^{\frac{5}{3},\frac{1}{2},e^\varrho}u_1(\varrho)+\frac{1}{20}D_{a^+}^{\frac{2}{3},\frac{1}{2},e^\varrho}u_1(\varrho)=\frac{1}{4(\varrho+2)^2}\frac{|u_1(\varrho)|}{1+|u_1(\varrho)|}+\frac{1}{32}\sin^2 y(\varrho)+1\\+\frac{1}{\sqrt{\varrho^2+1}},\quad\varrho\in[0,1],\\D_{a^+}^{\frac{7}{4},\frac{1}{3},e^\varrho}u_2(\varrho)+\frac{1}{10}D_{a^+}^{\frac{3}{4},\frac{1}{3},e^\varrho}u_2(\varrho)=\frac{\sin(2\pi u_2(\varrho))}{32\pi}+\frac{1}{2}+\frac{|u_2(\varrho)|}{16(1+|u_2(\varrho)|)},\quad\varrho\\\in[0,1],\\u_1(0)=0\qquad u_1(1)=\frac{1}{3}\ \text{ and } u_2(0)=0\\u_2(1)=\frac{1}{6}.\end{cases}\tag{5.1}$$

Here $\alpha_1=\frac{5}{3},\beta_1=\frac{1}{2},\alpha_2=\frac{7}{4},\beta_2=\frac{1}{3},\gamma_1=\gamma_2=\frac{11}{6},Q_1=\frac{1}{3},Q_2=\frac{1}{6},L_1=\frac{1}{20},L_2=\frac{1}{10}$. Set $\psi(\varrho)=e^\varrho$,

Example 5.1

$f_1(\varrho,u_1(\varrho),u_2(\varrho))=\frac{1}{4(\varrho+2)^2}\frac{|u_1(\varrho)|}{1+|u_1(\varrho)|}+\frac{1}{32}\sin^2 u_2(\varrho)+1+\frac{1}{\sqrt{\varrho^2+1}}$ and
$f_2(\varrho,u_1(\varrho),u_2(\varrho))=\frac{\sin(2\pi u_2(\varrho))}{32\pi}+\frac{1}{2}+\frac{|u_2(\varrho)|}{16(1+|u_2(\varrho)|)}$. Note that

$$|f_i(\varrho,u_1,u_2)|\leq\frac{\frac{1}{16}\max\{u_1,u_2\}}{1+|u_1|+|u_2|}$$

$$|f_i(\varrho,u_1,u_2)-f_i(\varrho,\widehat{u}_1,\widehat{u}_2)|\leq\frac{1}{16}(|u_1-\widehat{u}_1|+|u_2-\widehat{u}_2|).$$

Here $\mathscr{L}_i=\tau_i^*=\frac{1}{16}$. From the given data, we get $N_1\simeq 0.2$ and $N_2\simeq 0.4$. Clearly, the conditions (H_1) and (H_2) hold with $N=\max\{N_1,N_2\}=0.4>0$. Thus all the conditions of Theorem3.1 are satisfied. Therefore, system (1.2) has at least one solution on $[0,1]$. Moreover, we have

$\sigma_1\simeq 0.72$ and $\sigma_2\simeq 0.9$.

Thus, all conditions of Theorem3.2 are satisfied with $\sigma = \max\{\sigma_1, \sigma_2\} = 0.9 < 1$. Therefore, system (1.2) has a unique solution on $[0, 1]$. Finally, for $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$, we find that

$$\left|D_{a^+}^{\alpha_i,\beta_i,\psi}\widehat{u_i}(\varrho) + L_i D_{a^+}^{\alpha_i-1,\beta_i,\psi}\widehat{u_i}(\varrho) - f_i(\varrho, \widehat{u_1}(\varrho), \widehat{u_2}(\varrho))\right| \leq K\varepsilon$$

is satisfied. Then equation (4.2) is Ulam–Hyers stable with

$$\|(\widehat{u_1}, \widehat{u_2}) - (u_1, u_2)\|_{1-\gamma,\psi} \leq K\varepsilon, \qquad \varrho \in [0, 1],$$

where

$$K = 10 > 0.$$

6. Concluding remarks

The existence, uniqueness and Ulam–Hyers stability of solutions for a new coupled system of ψ-Hilfer sequential FDEs with two-point boundary conditions have obtained. Our investigations based on the reduction of FDEs to FIEs and applying the standard fixed point theorems due to Leray–Schauder and Banach. The acquired results in this paper are more general and cover many of the parallel problems that contain special cases of function ψ, because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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