ScienceDirect

AKCE International Journal of Graphs and Combinatorics

Volume 16, Issue 1, April 2019, Pages 8-17

Generalizations of supplemented lattices

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https://doi.org/10.1016/j.akcej.2018.02.005

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Abstract

Some generalizations of the concept of a supplemented lattice, namely a soc-supplemented-lattice, soc-amply-supplemented-lattice, soc-weak-supplemented-lattice, soc-$\oplus$-supplemented-lattice and completely soc-$\oplus$-supplemented-lattice are introduced. Various results are proved to show the relationship between these lattices. We have also proved that, if L is a soc-$\oplus$-supplemented-lattice satisfying the summand intersection property (SIP), then L is a completely soc-$\oplus$-supplemented-lattice.

Keywords

Modular lattice; Supplemented lattice; Soc-supplemented-lattice; Soc-amply-supplemented-lattice; Soc-weak-supplemented-lattice

1. Introduction

Mutlu[1], Tohidi[2], Wang and Ding[3], Wisbauer[4] and many others have studied the concept of a supplemented module and its generalizations. Let N and L be submodules of a module M. N is called a *supplement* of L if $N+L=M$ and N is minimal with respect to this property. A module M is called an *amply supplemented module* if for any two submodules A and B of M with $A+B=M$, B contains a supplement of A. A module M is called $\oplus$- *supplemented* if each submodule of M has a supplement that is a direct summand of M.

In 2012, Tohidi[2] introduced some generalizations of the concept of a supplemented module namely, a soc-supplemented-module, a soc-amply-supplemented-module, a soc-weak-supplemented-module, a soc-⊕-supplemented-module and a completely soc-⊕-supplemented-module. He proved various results to show relationship between these modules. He showed that, a direct summand of a soc-amply-supplemented-module is also a soc-amply-supplemented-module.

Călugăreanu[5] used lattice theory in module theory and studied several concepts from module theory in lattice theory. He introduced the concept of a supplement in terms of elements. Alizade and Toksoy[6] introduced the concepts of an ample supplement and an amply supplemented lattice in the context of a complete modular lattice. In[7] they also introduced the concepts of a weak supplement, a weakly supplemented lattice in the context of a complete modular lattice.

In this paper, we introduce the concepts of a soc-supplemented-lattice, a soc-amply-supplemented-lattice, a soc-weak-supplemented-lattice, a soc-⊕-supplemented-lattice and a completely soc-⊕-supplemented-lattice and obtain some results in the context of modular lattices.

Throughout in this paper L denotes a lattice. Wherever necessary we assume that $Soc\,(a)$ exists for any $a\in L$ and $Soc\,(L)=Soc\,(1)$.

2. Preliminaries

We recall some terms from lattice theory. These and undefined terms can be found in Grätzer[8].

Definition 1

A lattice L is called modular if for $a,b,c\in L$ with $a\leq c$, $a\vee(b\wedge c)=(a\vee b)\wedge c$.

Definition 2

If $a,b,c\in L$ are such that $a\vee b=c$ and $a\wedge b=0$ then we say that a,b are direct [illegible] we write $c=a\oplus b$. We say that c is a direct sum of a and b.

Definition 3

Let L be a lattice with 0. An element $a \in L$ is called an atom, if there does not exist any $b \in L$ such that $0 < b < a$.

Definition 4

A lattice L with 0 is said to be an atomistic lattice if every non-zero element $a \in L$ is the join of atoms of L contained in a.

Definition 5

[5, p. 47]

The join of all atoms of L, denoted by $Soc(L)$, is called the socle of the lattice L.

For $a \in L$, $Soc(a)$ is the socle of the lattice $[0, a]$.

We recall some definitions from Alizade and Toksoy[[6], [7]] and from Călugăreanu[5].

Definition 6

An element $a \in L$ is said to be small in L if $a \vee b \neq 1$ for every $b \neq 1$. We then write $a \ll L$.

Definition 7

An element $a \in L$ is called a supplement of an element $b \in L$ if $a \vee b = 1$ and a is minimal with respect to this property.

Lemma 1

Let L be a modular lattice and $a, b \in L$. a is a supplement of b in L if and only if $a \vee b = 1$ and $a \wedge b$ is small in $[0, a]$.

Proof

Suppose that a is a supplement of b in L. Then $a \vee b = 1$ and a is minimal with respect to this property. Let $(a \wedge b) \vee c = a$ for some $c \in [0, a]$, $c < a$. Then $1 = a \vee b = (a \wedge b) \vee c \vee b = b \vee c = 1$, a contradiction. Hence $a \wedge b$ is small in $[0, a]$.

Conversely, suppose that $a \vee b = 1$ and $a \wedge b$ is small in $[0, a]$. Let $c \vee b = 1$ for some $c < a$. We have $c \vee (b \wedge a) = (c \vee b) \wedge a = a$, a contradiction. Hence a is a supplement of b in L.

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Remark 1

The above equivalence does not hold in a nonmodular lattice.

Example 1

In the lattice shown in Fig. 1, $b \vee c = 1$ and $b \wedge c = 0$ is small in $[0, b]$ but b is not a supplement of c.

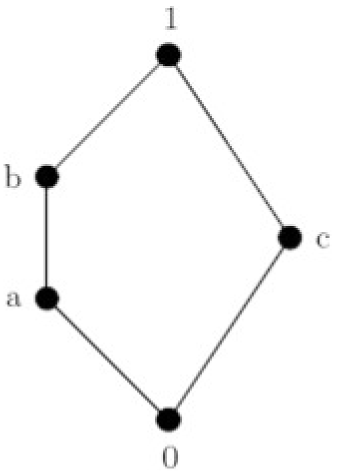

Fig. 1.

Definition 8

An element $a \in L$ is said to have ample supplements in L if for every element $b \in L$ with $a \vee b = 1$, $[0, b]$ contains a supplement of a in L.

A lattice L is said to be amply supplemented if every element $a \in L$ has ample supplements in L.

Definition 9

An element $a \in L$ is a weak supplement of $b \in L$ in L if and only if $a \vee b = 1$ and $a \wedge b \ll L$.

A lattice L is said to be weakly supplemented if every element $a \in L$ has a weak supplement in L.

3. Soc-s-lattices, soc-a-s-lattices and soc-w-s-lattices

In this section, L denotes a lattice with 0 and 1.

Let $a, b \in L$, $a \neq 0, 1$ and $b \neq 0, 1$ be such that $a \vee b = 1$, then b is called a soc-supplement of a in case $a \wedge b \leq Soc\,(b)$.

An element $a \in L$ is called a soc-supplement element if a is a soc-supplement of some element in L.

A lattice L is called a soc-supplemented lattice if every element of L has a soc-supplement in L. In short we say that L is a soc-s-lattice.

Example 2

Every complemented lattice is a soc-supplemented lattice.

Example 3

Let L be a finite lattice with only one atom and two dual atoms whose meet is different from that atom. Then L is not a soc-supplemented lattice.

Definition 11

A lattice L is called a soc-amply-supplemented lattice if $1 = a \vee b$, where $a, b \in L$ imply that a has a soc-supplement $c \in L$ such that $c \leq b$. In short we say that L is a soc-a-s-lattice.

An element $a \in L$ is called a soc-amply-supplemented element if $a = b \vee c$, where $b \leq a, c \leq a$ imply that b has a soc-supplement $d \leq a$ such that $d \leq c$. In short we say that a is a soc-a-s-element.

Let L be a lattice and $a \in L$. a is said to have a soc-ample-supplement in L if for any $b \in L$ with $a \vee b = 1$, a has a soc-supplement $c \in L$ such that $c \leq b$.

Example 4

Every atomistic complemented lattice is a soc-a-s-lattice.

Example 5

In the lattice L shown in Fig. 2, for elements $e, f \in L, 1 = e \vee f$, here $c \leq f$ but c is not a soc-supplement of e because $c \vee e \neq 1$. Hence L is not a soc-a-s-lattice.

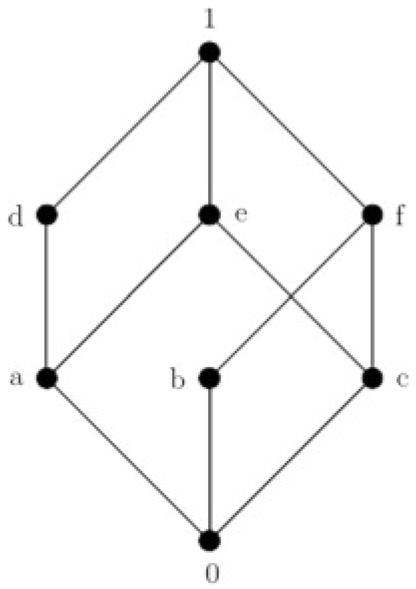

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Fig. 2.

The following two results are analogues of Proposition 2.1 and Lemma 2.2 from Tohidi[2].

Theorem 2

Let L be a modular soc-a-s-lattice and $a \in L$ be a direct summand of 1. Then a is a soc-a-s-element.

Proof

Let L be a soc-a-s-lattice and let a be a direct summand of 1. Then $a \oplus b = 1$ for some $b \in L$.

To show: a is a soc-a-s-element. Let $a = c \vee d$, where $c, d \in L$. Then $1 = (c \vee d) \vee b = c \vee (d \vee b) = c \vee (d \oplus b)$. Since L is a soc-a-s-lattice, there exists $f \in L$ such that $f \leq c$ with $f \vee (d \oplus b) = 1$ and $f \wedge (d \oplus b) \leq Soc\,(f)$. Now, by using modularity, we get

$$a = a \wedge [f \vee (d \vee b)] = f \vee [(d \vee b) \wedge a] = f \vee [d \vee (b \wedge a)] = f \vee d.$$

Also, $f \wedge d \leq f \wedge (d \oplus b) \leq Soc\,(f)$. Hence a is a soc-a-s-element.

Theorem 3

Let L be a modular lattice, $a, b \in L$ and a be a soc-s-element. If $a \vee b$ has a soc-supplement in L then so does b.

Proof

Suppose that $a \vee b$ has a soc-supplement say c in L. Then $(a \vee b) \vee c = 1$ and $(a \vee b) \wedge c \leq Soc\,(c)$. Since a is a soc-s-element and $(c \vee b) \wedge a \leq a$, there exists $d \in L$ such that $d \leq a$, $a = [(c \vee b) \wedge a] \vee d$ and $[(c \vee b) \wedge a] \wedge d \leq Soc\,(d)$. Then $(c \vee b) \wedge d \leq Soc\,(d)$. Now, by modularity, we get

$$1 = a \vee b \vee c = \{[(c \vee b) \wedge a] \vee d\} \vee b \vee c = \{a \wedge [(c \vee b) \vee d]\} \vee (b \vee c) = (b \vee c \vee a) \wedge (c \vee b \vee d) = (c \vee b) \vee d.$$

Thus, d is a soc-supplement of $c \vee b$ in L.

Claim

$c \vee d$ is a soc-supplement of b in L.

Clearly $(c \vee d) \vee b = 1$. We have $(d \vee b) \wedge c \leq (a \vee b) \wedge c \leq Soc\,(c)$. By modularity, we get

$$(c \vee d) \wedge b \leq (c \vee d) \wedge (d \vee b) \wedge (c \vee b) \leq (d \vee b) \wedge (c \vee d) \wedge (c \vee b)$$
$$\leq \{[(d \vee b) \wedge c] \vee d\} \wedge (c \vee b) \leq [c \wedge (d \vee b)] \vee [d \wedge (c \vee b)] \leq Soc\,(c) \vee Soc\,(d)$$
$$\leq Soc\,(c \vee d)\,.$$

Thus $c \vee d$ is a soc-supplement of b in L.

Theorem 4

Let L be a modular lattice and $a, b \in L$ be soc-supplemented elements. If $1 = a \oplus b$, then L is a soc-s-lattice.

Proof

Let $c \in L$ be such that $a \vee b \vee c = 1$ and $a \vee b \vee c$ has trivially a soc-supplement 0 in L. Then by Theorem 3, $b \vee c$ has a soc-supplement in L. Again by Theorem 3, c has a soc-supplement in L. Hence L is a soc-s-lattice.

The following result is analogue of Proposition 2.5 from Tohidi[2].

Theorem 5

Let L be a modular lattice and $a, b \in L$ be such that $a \vee b = 1$. If a and b have soc-ample-supplements in L then $a \wedge b$ also has a soc-ample-supplement in L.

Proof

Let $a, b \in L$ be such that $a \vee b = 1$. Suppose that a and b have soc-ample-supplements in L

To show: $a \wedge b$ has a soc-ample-supplement in L. Let $c \in L$ be such that $(a \wedge b) \vee c = 1$. Then $a = (a \wedge b) \vee (c \wedge a)$ and $b = (a \wedge b) \vee (c \wedge b)$. Therefore, $1 = a \vee (c \wedge b)$ and $1 = b \vee (c \wedge a)$. Since a and b have soc-ample-supplements in L, there exist $d, e \in L$ such that $d \leq c \wedge b$ and $e \leq c \wedge a$. Also $a \vee d = 1$, $a \wedge d \leq Soc(d)$ and $b \vee e = 1$, $b \wedge e \leq Soc(e)$. Now $d \leq c$ and $e \leq c$ implies that $d \vee e \leq c$. Now, $a = (a \wedge b) \vee e$ and $b = (b \wedge a) \vee d$. Therefore, $1 = (a \wedge b) \vee (e \vee d)$. Now, by modularity, we get

$$(e \vee d) \wedge (a \wedge b) \leq (e \vee d) \wedge (d \vee (a \wedge b)) \wedge (e \vee (a \wedge b)) \leq (d \vee (a \wedge b)) \wedge (e \vee d) \wedge (e \vee (a \wedge b)) \leq \{d \vee [e \wedge (d \vee (a \wedge b))]\} \wedge (e \vee (a \wedge b)) \leq [e \wedge (d \vee (a \wedge b))] \vee [d \wedge (e \vee (a \wedge b))] \leq [e \wedge (d \vee a) \wedge b] \vee [d \wedge (e \vee b) \wedge a] \leq [e \wedge 1 \wedge b] \vee [d \wedge 1 \wedge a] \leq (e \wedge b) \vee (d \wedge a) \leq Soc(e) \vee Soc(d) \leq Soc(e \vee d).$$

Hence $d \vee e$ is a soc-a-supplement of $a \wedge b$ in L.

The following result is an analogue of Theorem 2.6 from Tohidi[2].

Theorem 6

Let L be a modular lattice and $a \in L$. Then the following statements are equivalent.

(i) There is a decomposition $1 = b \oplus c$, where $b, c \in L$ with $b \leq a$ and $c \wedge a \leq Soc(c)$.

(ii) a has a soc-supplement $d \in L$ in L such that $d \wedge a$ is a direct summand of a.

Proof

$(i) \Rightarrow (ii)$ Let $1 = b \oplus d$ with $b \leq a$ and $d \wedge a \leq Soc(d)$. Then $a \vee d = 1$ and $a \wedge d \leq Soc(d)$ which means d is a soc-supplement of a in L.

We have $a = a \wedge 1 = a \wedge (b \vee d) = b \vee (d \wedge a)$ by using modularity. Also $b \wedge (d \wedge a) = 0$. Hence $d \wedge a$ is a direct summand of a.

$(ii) \Rightarrow (i)$ Suppose that d is a soc-supplement of a such that $a = b \oplus (d \wedge a)$. Then, $1 = a \vee d = b \vee (d \wedge a) \vee d = b \vee d$ and $b \wedge d = (b \wedge a) \wedge d = 0$. Hence b is a direct summand of 1.

Călugăreanu[5] developed the concept of an essential element in a lattice with least element 0.

Definition 12

[5, p. 39]

0. An element $a \in L$ is called an essential element if $a \wedge b \neq 0$, for any nonzero $b \in L$.

If a is essential in $[0, b]$ then we say that a is essential in b and write $a \leq_e b$ and call b as an essential extension of a.

If $a \leq_e b$ and there is no $c \in L$ such that $a \leq_e c$ and $b < c$, then we say that b is a maximal essential extension of a.

Theorem 7

Let L be a modular lattice, $a, b \in L$. Let b be a soc-supplement of a in L. If a is an essential element of L, then $a \wedge b = Soc(b)$ is a minimal essential element of $[0, b]$.

Proof

Let $0 \neq c \in L$ be such that $c \leq b$. Since a is essential in L, $a \wedge c \neq 0$, so $(a \wedge b) \wedge c \neq 0$. Thus $a \wedge b$ is an essential element in $[0, b]$. Now $Soc(b) \leq a \wedge b$. Since b is a soc-supplement of a in L, we have $1 = a \vee b$ and $a \wedge b \leq Soc(b)$. Thus $a \wedge b = Soc(b)$. Hence $Soc(b) = a \wedge b$ is a minimal essential element in $[0, b]$.

Theorem 8

Let L be a modular lattice. If every element in L is a soc-s-element, then L is a soc-a-s-lattice.

Proof

Let $a, b \in L$ be such that $a \vee b = 1$. We have $a \wedge b \leq b$ and since b is a soc-s-element. Let $c \in L$ be such that $c \leq b$, $b = (a \wedge b) \vee c$ and $(a \wedge b) \wedge c \leq Soc(c)$. Thus $a \wedge c \leq Soc(c)$. Also $1 = a \vee b = a \vee [(a \wedge b) \vee c] = a \vee c$. Thus $1 = a \vee c$ and $a \wedge c \leq Soc(c)$ imply L is a soc-a-s-lattice.

Definition 13

A lattice L is said to be a soc-weakly supplemented lattice if for any element $a \in L$, $a \neq 0, 1$ there exists $b \in L$ such that $a \vee b = 1$ and $a \wedge b \leq Soc(1)$. In short we say that L is a soc-w-s-lattice.

An element $a \in L$ is called soc-weak-supplement if a is a soc-weak-supplement of some element $b \in L$.

Example 6

Every complemented lattice is a soc-w-s-lattice.

Example 7

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In the lattice L shown in Fig. 3, for elements $f, h \in L$ such that $f \vee h = 1$ but $d = f \wedge h \nleq Soc(1) = a$ that is $d \nleq a$. Hence L is not a soc-w-s-lattice.

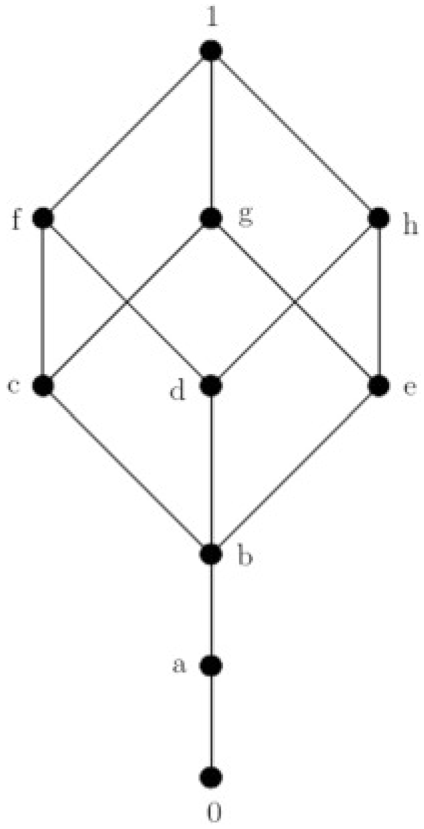

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Fig. 3.

The following lemma is an analogue of Proposition 9.8 from Anderson and Fuller[9].

Lemma 9

Let L and L' be two lattices and $f : L \to L'$ be a homomorphism satisfying $f(\vee_{i\in I} a_i) = \vee_{i\in I} f(a_i)$ then $f(Soc(1)) \leq Soc(1')$ for $1 \in L$ and $1' \in L'$.

Proof

For $1 \in L$ and $1' \in L'$, $Soc(1) = \vee(\text{all atoms of } L)$ and

$Soc(1') = \vee(\text{all atoms of } L')$. Now,

$f(Soc(1)) = f[\vee(\text{all atoms of } L)] = \vee[f(\text{all atoms of } L)] = Soc(1')$.

Hence, $f(Soc(1)) \leq Soc(1')$.

Typesetting math: 99% ›morphic image of a Soc-w-s-lattice is a Soc-w-s-lattice under a condition.

Theorem 10

Let L be a lattice satisfying $f(\vee_{i\in I} a_i) = \vee_{i\in I} f(a_i)$. Then any homomorphic image of a soc-w-s-lattice is a soc-w-s-lattice.

Proof

Let $f : L \to L'$ be an epimorphism and L be a soc-w-s-lattice. To show: L' is a soc-w-s-lattice. Let $a \in L'$ then $f^{-1}(a) \le L$. Since L is a soc-w-s-lattice, $f^{-1}(a)$ has a soc-weak-supplement $b \in L$, that means $L = f^{-1}(a) \vee b$ and $f^{-1}(a) \wedge b \le soc(1)$. Then $f\left(f^{-1}(a)\right) \vee f(b) = f(1) = 1' \in L'$ imply $1' = a \vee f(b)$. Now,

$$a \wedge f(b) = f\left(f^{-1}(a) \wedge b\right) \le f(Soc(1)) \le Soc(1') \ \ (\text{by Lemma 9}).$$

Thus $1' = a \vee f(b)$ and $a \wedge f(b) \le Soc(1')$ implies L' is a soc-w-s-lattice.

Lemma 11

Let L be an atomistic lattice and $a, b \in L$. If $a \le b$, then

$Soc(a) = a \wedge Soc(b)$.

Proof

Let $a \le b$. It is clear that $Soc(a) \le a \wedge Soc(b)$. Let $x \le a \wedge Soc(b)$. Since L is atomistic,

$x = \vee\{q_i : q_i$ is an atom of L and $q_i \le x\}$. Now, $x \le a$ implies $q_i \le a$ for all $q_i \le x$. Then $x \le \vee q_i \le Soc(a)$. Thus $Soc(a) = a \wedge Soc(b)$.

Theorem 12

Let L be an atomistic modular lattice. If L is a soc-w-s-lattice then every supplement element of L is a soc-w-s-element.

Proof

Suppose that $a \in L$ is a supplement in L. Since L is a soc-w-s-lattice, for any element $b \in L$ such that $b \le a$, there exists $c \in L$ such that $b \vee c = 1$ and $b \wedge c \le Soc(1)$. Now, by modularity, we get $a = a \wedge 1 = a \wedge [b \vee c] = b \vee [a \wedge c]$ and $b \wedge (a \wedge c) = a \wedge (b \wedge c) \le a \wedge Soc(1) = Soc(a) \le Soc(1)$ by Lemma 11. Thus $a = b \vee (a \wedge c)$ and $b \wedge (a \wedge c) \le Soc(1)$ imply a is a soc-w-s-element.

The following result is analogue of Lemma 2.18 from Tohidi[2].

Theorem 13

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Let L be a modular lattice, $a, b \in L$ and a be a soc-w-s-element. If $a \vee b$ has a soc-w-supplement in L, then so does b.

Proof

Let $a \vee b$ have a soc-w-supplement in L, then there exists $c \in L$ such that $(a \vee b) \vee c = 1$ and $(a \vee b) \wedge c \leq Soc(1)$. Since a is a soc-w-s-element and $(c \vee b) \wedge a \leq a$, there exists $d \in L$ such that $d \leq a$, $a = [(c \vee b) \wedge a] \vee d$ and $[(c \vee b) \wedge a] \wedge d \leq Soc(a)$ that is $(c \vee b) \wedge d \leq Soc(a)$. Now, by modularity, we get

$$1 = a \vee b \vee c = \{[(c \vee b) \wedge a] \vee d\} \vee b \vee c = \{a \wedge [(c \vee b) \vee d]\} \vee (b \vee c) = (b \vee c \vee a) \wedge (c \vee b \vee d) = (c \vee b) \vee d = (c \vee d) \vee b$$

and

$$(c \vee d) \wedge b \leq (c \vee d) \wedge (d \vee b) \wedge (c \vee b) \leq (d \vee b) \wedge (c \vee d) \wedge (c \vee b) \leq \{[(d \vee b) \wedge c] \vee d\} \wedge (c \vee b) \leq [c \wedge (d \vee b)] \vee [d \wedge (c \vee b)] \leq Soc(1) \vee Soc(a) \leq Soc(1).$$

Thus $c \vee d$ is a soc-w-supplement of b in L.

Theorem 14

Let L be a modular lattice and $1 = a \vee b$, $a, b \in L$. If a and b are soc-w-s-elements, then L is a soc-w-s-lattice.

Proof

Let $c \in L$ such that $a \vee b \vee c = 1$ and let $a \vee b \vee c$ have a soc-w-supplement 0 in L. Then by Theorem 13, $a \vee c$ has a soc-w-supplement in L. Again by Theorem 13, c has a soc-w-supplement in L. Hence L is soc-w-s-lattice.

Theorem 15

Every soc-a-s-lattice is a soc-s-lattice and every soc-s-lattice is a soc-w-s-lattice.

Proof

Let L be a soc-a-s-lattice. To show: L is a soc-s-lattice. Let $a, b \in L$ such that $a \vee b = 1$.

We claim: $a \wedge b \leq Soc(b)$.

Since L is a soc-a-s-lattice, there exists $c \in L$ such that $c \leq b$, $1 = a \vee c$ and $a \wedge c \leq Soc(c)$. Now, $a \wedge c \leq a \wedge b \leq Soc(c) \leq Soc(b)$. Thus $a \wedge b \leq Soc(b)$. Hence L is a soc-s-lattice.

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Next, let L be a soc-s-lattice. To prove: L is soc-w-s-lattice. Let $a \in L$. Since L is a soc-s-lattice, there exists $b \in L$ such that $a \vee b = 1$ and $a \wedge b \leq Soc(b)$. Now $a \wedge b \leq Soc(b) \leq Soc(1)$ that is $a \wedge b \leq Soc(1)$. Hence L is soc-w-s-lattice.

Remark 2

The following example shows that the converse of the above theorem need not be true.

Example 8

The lattice L shown in Fig. 2 is a soc-w-s-lattice but not a soc-a-s-lattice. Since, for $e, f \in L$, $1 = e \vee f$, here $c \leq f$ but c is not a soc-supplement of e because $c \vee e \neq 1$. Hence L is not a soc-a-s-lattice.

4. Soc-⊕-supplemented-lattices, completely soc-⊕-supplemented-lattices and lattices satisfying the summand intersection property

Definition 14

A lattice L is called a soc-⊕-supplemented-lattice if every element $a \in L$ has a soc-supplement $b \in L$ such that $1 = b \oplus c$, for some $c \in L$. In short we say that L is a soc-⊕-s-lattice.

Example 9

Every complemented lattice is a soc-⊕-s-lattice.

Example 10

In the lattice L shown in Fig. 4, $d \vee b = 1$. Here b is a soc-supplement of d, but b is not a direct summand of 1. Hence L is not a soc-⊕-s-lattice.

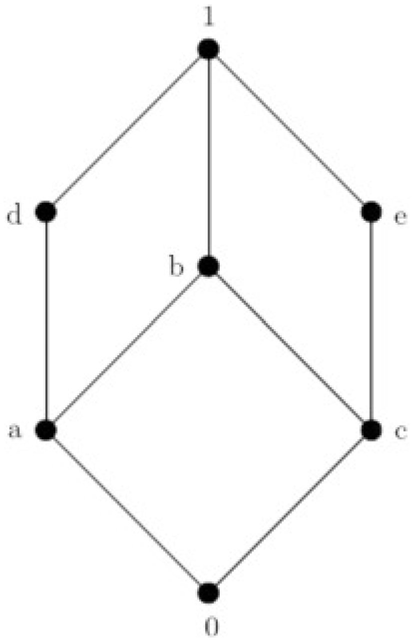

Fig. 4.

The following theorem is an analogue of Lemma 3.1 from Tohidi[2].

Theorem 16

Let L be a modular lattice and $a, b \in L$ be such that $a \vee b$ has a soc-supplement $c \in L$ in L and $a \wedge (b \vee c)$ has a soc-supplement $d \in L$ in a. Then $c \vee d$ is a soc-supplement of b in L.

Proof

Let c be a soc-supplement of $a \vee b$ in L and d be a soc-supplement of $a \wedge (b \vee c)$ in a. Then $(a \vee b) \vee c = 1$ with $(a \vee b) \wedge c \leq Soc(c)$ and $[a \wedge (b \vee c)] \vee d = a$ with $[a \wedge (b \vee c)] \wedge d \leq Soc(d)$ that is $(b \vee c) \wedge d \leq Soc(d)$. Now, by modularity, we get

$$1 = a \vee b \vee c = \{[(c \vee b) \wedge a] \vee d\} \vee b \vee c = \{a \wedge [(c \vee b) \vee d]\} \vee (b \vee c) = (b \vee c \vee a) \wedge (c \vee b \vee d) = c \vee b \vee d = b \vee (c \vee d)$$

and

$$(c \vee d) \wedge b \leq (c \vee d) \wedge (d \vee b) \wedge (c \vee b) \leq (d \vee b) \wedge (c \vee d) \wedge (c \vee b) \leq \{[(d \vee b) \wedge c] \vee d\} \wedge (c \vee b) \leq [c \wedge (d \vee b)] \vee [d \wedge (c \vee b)] \leq Soc(c) \vee Soc(d) \leq Soc(c \vee d).$$

Thus $1 = (c \vee d) \vee b$ and $(c \vee d) \wedge b \leq Soc(c \vee d)$ imply $c \vee d$ is a soc-supplement of b

Theorem 17

Let L be a modular lattice, $a, b \in L$ be soc- $\oplus$-s-elements and $1 = a \oplus b$. Then L is a soc- $\oplus$-s-lattice.

Proof

Let $c \in L$. Then $a \vee b \vee c = 1$ such that $a \vee b \vee c$ has trivially soc-supplement 0 in L. Let $d \in L$ be a soc-supplement of $b \wedge (a \vee c)$ in b, so that d is a direct summand of b. Then by Theorem 16, d is a soc-supplement of $a \vee c$ in L. Let e be a soc-supplement of $a \wedge (c \vee d)$ in a such that e is a direct summand of a. Again by Theorem 16, we have $d \vee e$ is a soc-supplement of c in L. Since d is a direct summand of b and e is a direct summand of a then $e \oplus d = e \vee d$ is a direct summand of 1. Hence L is a soc-$\oplus$-s-lattice.

Definition 15

A lattice L is said to be completely soc-$\oplus$-s-lattice if every direct summand of 1 other than an atom is a soc-$\oplus$-s-element.

Example 11

In the lattice L shown in Fig. 5, direct summands e and f of 1 which are not atoms are soc-$\oplus$-s-elements. For example, $e \in L$ with $e = a \vee b$ such that $a \wedge b = 0$, $Soc(b) = b$ and $0 \leq b$, here b is a soc-supplement of a in e which is a direct summand of e. Hence L is a completely soc-$\oplus$-s-lattice.

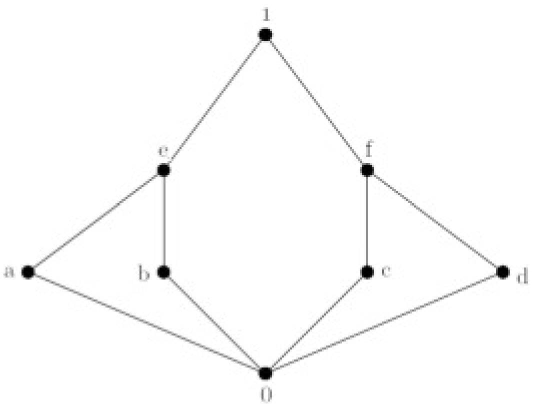

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Fig. 5.

Example 12

Typesetting math: 99% ɪ Fig. 4, direct summands d and e of 1 which are not atoms are not soc-$\oplus$-s-element because there is no such $w, x, y, z \in L$ such that $d = w \vee x$ and

$e = y \vee z$. Hence L is not a completely soc-$\oplus$-s-lattice.

The concept of the summand intersection property is known in module theory, see Akalan, Birkenmeier and Tercan[10].

The concept of the summand intersection property is also known in lattice theory, see Nimbhorkar and Shroff[11].

Definition 16

A lattice L satisfies Summand Intersection Property (SIP), if for any direct summands $a, b \in L$ of 1, $a \wedge b$ is also a direct summand of 1.

Theorem 18

Let L be a modular lattice. Suppose that L is a soc- $\oplus$-s-lattice satisfying SIP. Then L is a completely soc- $\oplus$-s-lattice.

Proof

Let $a \in L$ be a direct summand of 1. To show: a is a soc-$\oplus$-s-element. Let $b \leq a$. Since L is soc-$\oplus$-s-lattice, there exists a soc-supplement $c \in L$ of b such that $b \vee c = 1$, $b \wedge c \leq Soc(c)$ and $c \oplus d = 1$. Now, by modularity, we get $a = a \wedge 1 = a \wedge (b \vee c) = b \vee (c \wedge a)$. Since L satisfies the property SIP, $a \wedge c$ is a direct summand of 1. So $b \wedge (a \wedge c) = b \wedge c$, $b \wedge c \leq Soc(c) \leq Soc(1)$ that is $b \wedge c \leq Soc(1)$ and $b \wedge c \leq a \wedge c$. Therefore, $b \wedge c \leq (a \wedge c) \wedge Soc(1) = Soc(a \wedge c)$. Thus $a \wedge c$ is a soc-supplement of b in a which is a direct summand of a. Hence a is a soc- $\oplus$ -s-lattice.

Acknowledgements

The authors are thankful to the referees for fruitful suggestions, which improved the paper.

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Peer review under responsibility of Kalasalingam University.