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Results in Applied Mathematics
Volume 7, August 2020, 100115

Ulam–Hyers–Mittag-Leffler stability for a ψ-Hilfer problem with fractional order and infinite delay

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https://doi.org/10.1016/j.rinam.2020.100115
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Abstract

The research article concerned with developing the qualitative theory for nonlinear fractional functional differential equations (FFDEs) of arbitrary order with infinite delay involving generalized Hilfer fractional derivative of the form:

$$\begin{cases} {}^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} y(t) = f(t, y_t); \quad t \in (0, b], \\ \mathscr{I}_{0^+}^{1-\gamma;\psi} y(0^+) = y_0, \\ y(t) = \varphi(t); \qquad t \in (-\infty, 0]. \end{cases}$$

Here $0 < \alpha < 1, 0 \leq \beta \leq 1, y_0 \in \mathbb{R}$, and $\mathscr{D}_{0^+}^{\alpha,\beta;\psi}$, $\mathscr{I}_{0^+}^{1-\gamma;\psi}$ are generalized fractional operators in the concepts Hilfer and Riemann–Liouville, respectively. Some new and recent results of existence and Ulam–Hyers–Mittag-Leffler (UHML) stability of solution for the proposed problem will also be highlighted. The concerned analysis is carried out via using the Banach fixed point theorem, Picard operator method, and generalized Gronwall's inequality. Finally, an example is given to illustrate the effectiveness of our main results.

Previous Next

MSC

34K37; 26A33; 70G10; 34A12; 47H10

Keywords

Fractional differential equations; Phase space; Existence and stability; Picard operator; Fixed point theorem

1. Introduction

Over the last years, the stability results of functional differential equations have been strongly developed. Very significant contributions about this topic were introduced by Ulam [1], Hyers [2] and this type of stability called Ulam–Hyers (UH) stability. Thereafter improvement of UH stability provided by Rassias [3] in 1978. For some recent results of stability analysis by different types of fractional derivative operator, we refer the reader to a series of papers [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14].

On the other hand, the Ulam-type stability of delay differential equations (DDEs) was investigated in [15], [16]. In [15], the results for a DDE were obtained using the Picard operator method, and in [16] the authors adopted a similar approach to establish the existence and uniqueness results for a Caputo-type fractional-order DDEs. In the same context [17] the author discussed the existence and uniqueness of solutions and UH and UH–Rassias stabilities for ψ-Hilfer nonlinear fractional differential equations (FDEs) via a generalized Gronwall inequality. Sousa and Oliveira [8] introduced a ψ-Hilfer fractional derivative and established new results to ψ-Hilfer FDEs that generalizes some of the studies reported in the literature [18], [19]. There were many results on the existence, uniqueness, and stability solutions for the non-linear FDEs can be found in the articles [20], [21], [22], [23], [24] and the references mentioned in them.

For the recent review of FFDEs, we will survey some of the works as follows:

D. Otrocol, V. Ilea in [15] studied the UH stability and generalized UH–Rassias stability for the following DDE

$$\begin{cases} u'(t) = f(t, u(t), u(h(t)));\quad t \in [a,b], \\ u(t) = \psi(t);\qquad t \in [a-h, a]. \end{cases}$$

J. Wang and Y. Zhang [16], proved some results of existence, uniqueness, and UHML stability of Caputo-type FFDE

$$\begin{cases} {}^{C}\mathscr{D}_{0^+}^{\alpha} u(t) = f(t, u(t), u(h(t))); & t \in (0, d], \\ u(t) = \psi(t); & t \in [-h, 0]. \end{cases}$$

Liu etal. in [25] established the existence, uniqueness, and UHML stability of solutions to a class of ψ-Hilfer FFDE

$$\begin{cases} {}^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} u(t) = f(t, u(t), u(h(t))); & t \in (0, d], \\ I_{0^+}^{1-\gamma;\psi} u(0^+) = u_0 \in \mathbb{R}, \\ u(t) = \varphi(t); & t \in [-h, 0]. \end{cases}$$

K.D. Kucche, and P.U. Shikhare in [26] studied the existence, uniqueness of a solution and Ulam type stabilities for Volterra delay integro-differential equations on a finite interval

$$\begin{cases} u'(t) = f\left(t, y(t), y(g(t)), \int_0^t h(t, s, y(s), y(g(s)))\, ds\right); & t \in [0, b], \\ u(t) = \varphi(t); \quad t \in [-r, 0], \quad 0 < r < \infty. \end{cases} \tag{1.1}$$

Motivated and inspired by the aforementioned work, in this article, we prove the existence, uniqueness and UHML stability of solutions to a class of for ψ-Hilfer problem of FFDEs with infinite delay

$$\begin{cases} {}^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} y(t) = f(t, y_t); & t \in (0, b], \\ \mathscr{I}_{0^+}^{1-\gamma;\psi} y(0^+) = y_0 \in \mathbb{R}, \\ y(t) = \varphi(t); & t \in (-\infty, 0]. \end{cases} \tag{1.2}$$

Here $\mathscr{D}_{0^+}^{\alpha,\beta;\psi}(.)$ and $\mathscr{I}_{0^+}^{1-\gamma;\psi}(.)$ are ψ—Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$, and ψ— Riemann-Liouville (R-L) fractional integral of order $1-\gamma$ $(\gamma = \alpha + \beta(1-\alpha))$ respectively, $\varphi \in \mathscr{B}$ (phase space) and $f : (0, b] \times \mathscr{B} \to \mathbb{R}$ is a given function. For each y defined on $(-\infty, b]$ and for any $t \in [0, b]$ we denote by y_t the element of $\mathscr{B}$ defined by $y_t(s) = y(t+s), \quad -\infty < s \le 0$, where $y_t(\cdot)$ represent the history of the state from time $-\infty$ up to time t.

The main contributions are highlighted as follows:

- The obtained results more general of previous studies [15], [16], [25].
- The advantages of the problem considered and the importance of obtained results have been provided in the introduction section. It is the first work concerning fractional functional differential equations with infinite delay involving ψ-Hilfer fractional derivative.
- The formula provided for the solution to FFDEs (1.2) includes the formula for solutions of FFDEs involving Riemann–Liouville and Caputo fractional derivatives.

- The existence of a unique solution and the stability of UHML are proved in phase space $\mathscr{B}$ by means of the Picard operator method and Banach fixed point theorem and generalized Gronwall's inequality.

The rest of the structured this paper as follows. In Section 2, we will briefly recall some basic definitions and the results that are applied throughout the paper. Section 3 studies the existence, uniqueness and UHML stability results on the ψ-Hilfer problem of FFDE (1.2) via using the Banach fixed point theorem, Picard operator method, and generalized Gronwall's inequality. At the end, an example is included to illustrate the applicability of the obtained results.

2. Preliminaries

In this section, we recall the basic definitions and results which related to ψ- fractional calculus (ψ-R–L fractional integral and derivative, ψ-Caputo fractional derivative, and ψ-Hilfer fractional derivative) and some results of nonlinear analysis (Picard operator method, fixed point theorems, and generalized Gronwall's inequality). Let $I=(-\infty,b]$ $(b>0)$, $I_1=[0,b]$, $I_2=(0,b]$ and $I^*=(-\infty,0]$, and Let $C(I_1,\mathbb{R})$ and $C^n(I_1,\mathbb{R})$ be the Banach spaces of continuous functions, n-times continuously differentiable functions on I_1, respectively. Moreover, for any $h\in C(I_1,\mathbb{R})$, we have $\|h\|_C=\max\{|h(t)|:t\in I_1\}$. On the other hand, we have n-times absolutely continuous functions given by

$$AC^n(I_1,\mathbb{R})=\left\{h:I_1\to\mathbb{R}:h^{(n-1)}(t)\in AC(I_1,\mathbb{R})\right\},$$

where $AC(I_1,\mathbb{R})$ is the space of functions which are absolutely continuous on I_1. The weighted spaces $C_{1-\gamma;\psi}(I_1,\mathbb{R})$ and $C^n_{1-\gamma;\psi}(I_1,\mathbb{R})$ are defined by (see[8])

$$C_{1-\gamma;\psi}(I_1,\mathbb{R})=\left\{h:I_2\to\mathbb{R};\quad[\psi(t)-\psi(0)]^{1-\gamma}h(t)\in C(I_1,\mathbb{R})\right\},$$
$$C^n_{1-\gamma;\psi}(I_1,\mathbb{R})=\left\{h:I_2\to\mathbb{R}:h(t)\in C^{n-1}(I_1,\mathbb{R});\right.$$
$$\left.h^{(n)}(t)\in C_{1-\gamma;\psi}(I_1,\mathbb{R})\right\},$$

where $0\le\gamma<1$, with the norms

$$\|h\|_{C_{1-\gamma;\psi}}=\max_{t\in I_1}\left|[\psi(t)-\psi(a)]^{1-\gamma}h(t)\right|,$$
$$\|h\|_{C^n_{1-\gamma;\psi}}=\sum_{k=0}^{n-1}\|h^{(k)}\|_C+\|h^{(n)}\|_{C_{1-\gamma;\psi}}.$$

respectively. In particular, if $n=0$, we have $C^0_{1-\gamma;\psi}(I_1,\mathbb{R})=C_{1-\gamma;\psi}(I_1,\mathbb{R})$.

Definition 2.1

[27]

Let $\alpha > 0$ and ψ be an increasing function, having a continuous derivative ψ' on I_1. Then the left-sided R–L fractional integral of order α for an integrable function $h : I_1 \to \mathbb{R}$ with respect to ψ on I_1 is defined by

$$\mathscr{I}_{0^+}^{\alpha,\psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s)\, ds, \quad t > 0.$$

Definition 2.2

[27]

Let $0 < \alpha < 1$, $h \in C(I_1, \mathbb{R})$, and $\psi \in C^1(I_1, \mathbb{R})$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in I_1$. Then the left sided R–L fractional derivative of h of order α with respect to ψ is given by

$$\mathscr{D}_{0^+}^{\alpha,\psi} h(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) \mathscr{I}_{0^+}^{1-\alpha,\psi} h(t)$$
$$= \frac{1}{\Gamma(1-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) \int_0^t \psi'(s)(\psi(t) - \psi(s))^{-\alpha} h(s)\, ds, \quad t > 0,$$

provided the right side is piecewise continuous defined on I_1.

Definition 2.3

[28]

Let $0 < \alpha < 1$, and $h \in AC^1(I_1, \mathbb{R})$, and let ψ is defined as in Definition2.2. The left-sided Caputo fractional derivative of a function h of order α with respect to ψ is defined by

$${}^{C}\mathscr{D}_{0^+}^{\alpha,\psi} h(t) = \mathscr{I}_{0^+}^{1-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) h(t)$$
$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{-\alpha} h_\psi^{(1)}(s)\, ds, \quad t > 0,$$

where $h_\psi^{(1)}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) h(t)$.

Definition 2.4

[8]

Let $0 < \alpha < 1$, and $f, \psi \in C^1(I_1, \mathbb{R})$ be two functions and ψ is defined as in Definition2.2. The left-sided Hilfer fractional derivative of a function h of order α and type $0 \leq \beta \leq 1$ with respect to ψ is defined by

$${}^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} h(t) = \mathscr{I}_{0^+}^{\beta(1-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) \mathscr{I}_{0^+}^{(1-\beta)(1-\alpha);\psi} h(t); \quad t > 0.$$

One has,

$$^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}h(t)=\mathscr{I}_{0^+}^{\beta(1-\alpha);\psi}\mathscr{D}_{0^+}^{\gamma;\psi}h(t);$$

where

$$\mathscr{D}_{0^+}^{\gamma;\psi}h(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathscr{I}_{0^+}^{(1-\beta)(1-\alpha);\psi}h(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathscr{I}_{0^+}^{1-\gamma;\psi}h(t).$$

Remark 2.5

We note that, if $\psi(t)=t$ in Definitions Definition 2.1, Definition 2.2, Definition 2.3, then $\mathscr{I}_{0^+}^{\alpha,\psi}$, $\mathscr{D}_{0^+}^{\alpha,\psi}$ and $^{C}\mathscr{D}_{0^+}^{\alpha,\psi}$ reduces to the classical fractional integral and derivative.

Remark 2.6

From Definition 2.4, we observe that:

(i)

If $\psi(t)=t$, the operator $^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}$ can be written as

$$^{H}\mathscr{D}_{0^+}^{\alpha,\beta}=\mathscr{I}_{0^+}^{\beta(1-\alpha)}\mathscr{D}\mathscr{I}_{0^+}^{(1-\gamma)}=\mathscr{I}_{0^+}^{\beta(1-\alpha)}D_{0^+}^{\gamma},\qquad \gamma=\alpha+\beta(1-\alpha);$$
$$\mathscr{D}=\frac{d}{dt}.$$

(ii)

If $\beta=0$, then ψ-Hilfer fractional derivative $^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}$ reduces to $\mathscr{D}_{0^+}^{\alpha;\psi}$, i.e. $^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}=\mathscr{D}_{0^+}^{\alpha;\psi}$.

(iii)

If $\beta=1$, then ψ-Hilfer fractional derivative $^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}$ reduces to $^{C}\mathscr{D}_{0^+}^{\alpha;\psi}$, i.e. $^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}={}^{C}\mathscr{D}_{0^+}^{\alpha;\psi}$.

In the forthcoming analysis, we need the following weighted space:

$$C_{1-\gamma;\psi}^{\gamma}(I_1,\mathbb{R})=\left\{h\in C_{1-\gamma;\psi}(I_1,\mathbb{R});\quad \mathscr{D}_{0^+}^{\gamma;\psi}h\in C_{1-\gamma;\psi}(I_1,\mathbb{R})\right\},\tag{2.1}$$

where $0\le\gamma<1$.

Lemma 2.7

[8]

Let $\gamma=\alpha+\beta(1-\alpha)$ *where* $0<\alpha<1$ *and* $0\le\beta\le1$. *Then, if* $h\in C_{1-\gamma;\psi}^{\gamma}(I_1,\mathbb{R})$ *we have*

$$\mathscr{I}_{0^+}^{\gamma;\psi}\mathscr{D}_{0^+}^{\gamma;\psi}h=\mathscr{I}_{0^+}^{\alpha;\psi}\ {}^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}h$$

and

$$\mathscr{D}_{0^+}^{\gamma;\psi}\ \mathscr{I}_{0^+}^{\alpha;\psi}h=\mathscr{D}_{0^+}^{\beta(1-\alpha);\psi}h.$$

Lemma 2.8

[8]

Let $0<\alpha<1, 0\le\beta\le1$, *and* $h\in C^1(I_1,\mathbb{R})$. *Then we have*

$$ {}^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\ \mathscr{I}_{0^+}^{\alpha;\psi}h(t)=h(t). $$

Lemma 2.9

[27]

Let $\alpha,\sigma>0$. *Then we have*

$$ \mathscr{I}_{0^+}^{\alpha;\psi}[\psi(t)-\psi(0)]^{\sigma-1}=\frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)}[\psi(t)-\psi(0)]^{\alpha+\sigma-1}, $$

and

$$ \mathscr{D}_{0^+}^{\alpha;\psi}[\psi(t)-\psi(0)]^{\alpha-1}=0,\quad 0<\alpha<1. $$

Lemma 2.10

[28]

For $\alpha>0$ *and* $\lambda\in\mathbb{R}\setminus\{0\}$, *we have*

$$ \mathscr{I}_{0^+}^{\alpha;\psi}E_\alpha[\lambda(\psi(t)-\psi(0))^\alpha]=\frac{1}{\lambda}\left(E_\alpha\left[\lambda(\psi(t)-\psi(0))^\alpha\right]-1\right). $$

Lemma 2.11

[8]

Let $\gamma=\alpha+\beta(1-\alpha)$ *where* $0<\alpha<1$ *and* $0\le\beta\le1$. *Then, if* $h\in C_{1-\gamma;\psi}(I_1,\mathbb{R})$ *and* $\mathscr{I}_{0^+}^{1-\gamma;\psi}h\in C^1_{1-\gamma;\psi}(I_1,\mathbb{R})$ *we have*

$$ \mathscr{I}_{0^+}^{\alpha;\psi}\ {}^{H}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}h(t)=h(t)-\frac{\mathscr{I}_{0^+}^{(1-\beta)(1-\alpha);\psi}h(0)}{\Gamma(\alpha+\beta(1-\alpha))}(\psi(t)-\psi(0))^{(1-\beta)(\alpha-1)} $$
$$ =h(t)-\frac{\mathscr{I}_{0^+}^{1-\gamma;\psi}h(a)}{\Gamma(\gamma)}(\psi(t)-\psi(0))^{\gamma-1}. $$

Lemma 2.12

[8]

Let $h \in C_\gamma(I_1, \mathbb{R})$, *and* $0 < \gamma < \alpha < 1$. *Then we have*

$$\mathscr{I}_{0^+}^{\alpha;\psi} h(0) = \lim_{t \longrightarrow 0^+} \mathscr{I}_{0^+}^{\alpha;\psi} h(t) = 0.$$

Lemma 2.13

[17]

If $\alpha > 0$ *and* $0 \leq \gamma < 1$, *then* $\mathscr{I}_{0^+}^{\alpha;\psi}(\cdot)$ *is bounded from* $C_\gamma(I_1, \mathbb{R})$ *to* $C_\gamma(I_1, \mathbb{R})$. *Further, if* $\gamma \leq \alpha$, *then* $\mathscr{I}_{0^+}^{\alpha;\psi}(\cdot)$ *is bounded from* $C_\gamma(I_1, \mathbb{R})$ *to* $C(I_1, \mathbb{R})$.

Definition 2.14

[16]

Let (Y, d) be a metric space. Now $T : Y \longrightarrow Y$ is a Picard operator if there exists $y^* \in Y$ such that $F_T = y^*$ where $F_T = \{y \in Y : T(y) = y\}$ is the fixed point set of T, and the sequence $(T^n(y_0))_{n \in \mathbb{N}}$ converges to y^* for all $y_0 \in Y$.

Definition 2.15

[29], [30]

A linear topological space of functions from $(-\infty, 0]$ into $\mathbb{R}$, with seminorm $\|\cdot\|_{\mathscr{B}}$, is called an admissible phase space if $\mathscr{B}$ has the following properties:

(H1)

If $y : (-\infty, b] \to \mathbb{R}$ is continuous on $[0, b]$ and $y_0 \in \mathscr{B}$, then for every $t \in [0, b]$ the following conditions hold:

(i) $y_t \in \mathscr{B}$;

(ii) $|y(t)| \leq \mathscr{H} \|y_t\|_{\mathscr{B}}$, where $\mathscr{H} > 0$ is a constant, and $|\varphi(0)| \leq \mathscr{H} \|\varphi\|_{\mathscr{B}}$ for all $\varphi \in \mathscr{B}$.

(iii) $\|y_t\|_{\mathscr{B}} \leq K(t) \sup_{0 \leq s \leq t} |y(s)| + M(t) \|y_0\|_{\mathscr{B}}$, where $K, M : [0, +\infty) \to [0, +\infty)$ with K continuous and M locally bounded, such that K, M are independent of $y(.)$. Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

(H2)

For the function $y(.)$ in (**H1**), the function $t \to y_t$ is continuous from $[0, b]$ into $\mathscr{B}$.

(H3)

The space $\mathscr{B}$ is complete.

Lemma 2.16

[16]

Let $(Y, d, \leq)$ be an ordered metric space, and let $T : Y \longrightarrow Y$ be an increasing Picard operator with $F_T = \{y_T^*\}$. Then for $y \in Y$, $y \leq T(y)$ implies $y \leq y_T^*$, while $y \geq T(y)$ implies $y \geq y_T^*$.

Lemma 2.17

[6]

Let $f : I_2 \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the ψ-Hilfer problem

$$\begin{aligned} &{}^H\mathscr{D}_{0^+}^{\alpha,\beta;\psi} y(t) = f(t, y(t));\;\; t \in I_2 = (0, b],\\ &\mathscr{I}_{0^+}^{1-\gamma;\psi} y(0) = y_0, \end{aligned}$$

is equivalent to the integral equation

$$y(t) = \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, y(s))\, ds \tag{2.2}$$

.

Lemma 2.18

[17], Generalized Gronwall's Inequality

Let u, v, be two integrable functions and h continuous, with domain $[a, b]$. Let $\psi \in C[a, b]$ an increasing function such that $\psi'(t) \neq 0, \forall t \in [a, b]$. Assume that u and v are nonnegative and h is nonnegative and nondecreasing. If

$$u(t) \leq v(t) + h(t) \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s)\, ds,$$

then, for all $t \in [a, b]$,we have

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(s)(\psi(t) - \psi(s))^{\alpha k-1} v(s)\, ds. \tag{2.3}$$

Further, if v is a nondecreasing function on $[a, b]$ then

$$u(t) \leq v(t) E_\alpha\left(h(t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha\right),$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function with one parameter α.

3. Main results

In this section, we investigate the existence, uniqueness, and UHML stability results on the ψ-Hilfer problem of FFDE (1.2).

3.1. An existence and uniqueness results

Let us begin by introducing the following space:

$$\Omega_{\mathscr{B},\gamma,\psi} = \left\{ y : I \longrightarrow \mathbb{R} : y|_{I^*} \in \mathscr{B}, y|_{I_1} \in C_{1-\gamma;\psi}(I_1, \mathbb{R}) \right\}.$$

It is easy to verify that $\Omega_{\mathscr{B},\gamma,\psi}$ is a Banach space with respect to the norm

$$\|y\|_{\Omega_{\mathscr{B},\gamma,\psi}} = \|y_0\|_{\mathscr{B}} + \|y\|_{C_{1-\gamma;\psi}}.$$

To establish our results, we need the following hypotheses:

$(\mathbf{F}_1)$

$f : I_2 \times \mathscr{B} \to \mathbb{R}$ is continuous and there exists constant $M_f > 0$ such that

$$|f(t,u) - f(t,v)| \le M_f \quad \|u - v\|_{\mathscr{B}}, \forall \quad t \in I_2, \quad u, v \in \mathscr{B}.$$

$(\mathbf{F}_2)$

The following inequality holds

$$\frac{(\psi(b)-\psi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} M_f K_b < 1,$$

where $K_b = \sup\{|K(t)| : t \in I_1\}$ and $K(t)$ defined as in **(H1).**

Definition 3.1

Let $\gamma = \alpha + \beta(1-\alpha)$ where $0 < \alpha < 1, and \quad 0 \le \beta \le 1$. Then $y \in \Omega_{\mathscr{B},\gamma,\psi}$ defined by

$$y(t) = \begin{cases} \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, y_s)\, ds, \qquad t \in I_2, \\ \varphi(t); \qquad t \in I^* \end{cases} \tag{3.1}$$

is said to be a solution of (1.2) if y satisfies the equation ${}^H\mathscr{D}_{0^+}^{\alpha,\beta;\psi} y(t) = f(t, y_t)$, $t \in I_2$, with the conditions $\mathscr{I}_{0^+}^{1-\gamma;\psi} y(0) = y_0$, $y(t) = \varphi(t)$ for $t \in I^*$ and $y|_{I_1} \in C_{1-\gamma;\psi}(I_1, \mathbb{R})$.

Theorem 3.2

Assume that $(\mathbf{F}_1)$ *and* $(\mathbf{F}_2)$ *are satisfied. Then* ψ*-Hilfer problem of FFDE* (1.2) *has a unique solution in* $\Omega_{\mathscr{B},\gamma,\psi}$.

Proof

Transform the problem (1.2) into a fixed point problem. Define the operator $T_f : \Omega_{\mathscr{B},\gamma,\psi} \longrightarrow \Omega_{\mathscr{B},\gamma,\psi}$ by

$$(T_f y)(t) = \begin{cases} \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, y_s)\, ds, \qquad t \in I_2, \\ \varphi(t); \qquad t \in I^*. \end{cases} \tag{3.2}$$

For any function $\varphi \in \mathscr{B}$, we define $\tilde{\varphi} : I \to \mathbb{R}$ by

$$\tilde{\varphi}(t) = \begin{cases} 0, & t \in I_2, \\ \varphi(t); & t \in I^*, \end{cases} \tag{3.3}$$

and then $\tilde{\varphi}(0) = \varphi(0)$. For each function $z \in C_{1-\gamma;\psi}(I_1, \mathbb{R})$ with $z(0) = 0$ we denote by $\tilde{z} : I \to \mathbb{R}$ the function defined by

$$\tilde{z}(t) = \begin{cases} (\psi(t)-\psi(0))^{1-\gamma} z(t); & t \in I_2, \\ 0, & t \in I^*. \end{cases}$$

If $y(.)$ satisfies the integral equation

$$y(t) = \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, y_s)\, ds,$$

for $t \in I_2$, with $y(t) = \varphi(t)$, for $t \in I^*$, then we can analyze $y(.)$ as follows $y(t) = \tilde{\varphi}(t) + z(t); t \in I_2$. It is easy to see that $y_t = \tilde{\varphi}_t + \tilde{z}_t$, for every $t \in I_2$ if and only if z satisfies $z_0 = 0$ and

$$z(t) = \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \varphi(0) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, \tilde{\varphi}_s + \tilde{z}_s)\, ds, \tag{3.4}$$

for $t \in I_2$, with $\tilde{z}_0 = 0$. Set $\Omega^*_{\mathscr{B},\gamma,\psi} = \{z \in \Omega_{\mathscr{B},\gamma,\psi}, z_0 = 0\}$. For $z \in \Omega^*_{\mathscr{B},\gamma,\psi}$, let $\|.\|_{\Omega^*_{\mathscr{B},\gamma,\psi}}$ be seminorm in $\Omega^*_{\mathscr{B},\gamma,\psi}$ defined by

$$\|z\|_{\Omega^*_{\mathscr{B},\gamma,\psi}} = \|z_0\|_{\mathscr{B}} + \|z\|_{C_{1-\gamma;\psi}} \\ = \sup\left\{(\psi(t)-\psi(0))^{1-\gamma} |z(t)| : t \in I_2\right\}. \tag{3.5}$$

Since $\left(\Omega^*_{\mathscr{B},\gamma,\psi}, \|z\|_{\Omega^*_{\mathscr{B},\gamma,\psi}}\right)$ is a Banach space for $z \in \Omega^*_{\mathscr{B},\gamma,\psi}$, we define the operator $T^*_f : \Omega^*_{\mathscr{B},\gamma,\psi} \to \Omega^*_{\mathscr{B},\gamma,\psi}$ by

$$\left(T^*_f z\right)(t) = \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \varphi(0) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s, \tilde{\varphi}_s + \tilde{z}_s)\, ds, \tag{3.6}$$

for $t \in I_2$, and $\left(T^*_f z\right)(t) = 0$, for $t \in I^*$. Then, we get $\left(T^*_f z\right)_0 = 0$. Obviously the operator T_f having a unique fixed point is equivalent to T^*_f having one. So, we will show that the operator T^*_f has a unique fixed point. Note that for any continuous function f, the operator T^*_f is also continuous. Indeed, let us set $sup_{s\in I_2}|f(s,0)|=\mu<\infty$, and for all $t,t+\epsilon\in I_2$, we have

$$\left|T_f^*(z)(t+\epsilon)-T_f^*(z)(t)\right|=\left|\frac{(\psi(t+\epsilon)-\psi(0))^{\gamma-1}-(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\varphi(0)\right.$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{t+\epsilon}\psi'(s)(\psi(t+\epsilon)-\psi(s))^{\alpha-1}f(s,\tilde{\varphi}_s+\tilde{z}_s)ds$$
$$\left.-\frac{1}{\Gamma(\alpha)}\int_0^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,\tilde{\varphi}_s+\tilde{z}_s)ds\right|$$
$$\leq\left|\frac{(\psi(t+\epsilon)-\psi(0))^{\gamma-1}-(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\varphi(0)\right.$$
$$+$$
$$\frac{1}{\Gamma(\alpha)}\int_0^{t+\epsilon}\psi'(s)(\psi(t+\epsilon)-\psi(s))^{\alpha-1}(|f(s,\tilde{\varphi}_s+\tilde{z}_s)-f(s,0)|+|f(s,0)|)ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}$$
$$\left.\times(|f(s,\tilde{\varphi}_s+\tilde{z}_s)-f(s,0)|+|f(s,0)|)ds\right|$$
$$\leq\left|\frac{(\psi(t+\epsilon)-\psi(0))^{\gamma-1}-(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\varphi(0)\right.$$
$$\left.+\left(\frac{(\psi(t+\epsilon)-\psi(0))^{\alpha}-(\psi(t)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M_fK_b\|z\|_{\Omega^*_{\mathscr{B},\gamma,\psi}}+\mu\right)\right|,$$

where we used the fact $\|\tilde{\varphi}_s+\tilde{z}_s\|_{\mathscr{B}}\leq K_b\|z\|_{\Omega^*_{\mathscr{B},\gamma,\psi}}$ due to (**H1**). Thus we observe that

$$\left|T_f^*(z)(t+\epsilon)-T_f^*(z)(t)\right|\to 0,\quad \epsilon\to 0.$$

Next, we show $T_f^*:\Omega^*_{\mathscr{B},\gamma,\psi}\to\Omega^*_{\mathscr{B},\gamma,\psi}$ defined by (3.6) is a contraction mapping in $\Omega^*_{\mathscr{B},\gamma,\psi}$.

Let us consider $z,z^*\in\Omega^*_{\mathscr{B},\gamma,\psi}$ and for each $t\in I_2$, we have

$$\left|[\psi(t)-\psi(0)]^{1-\gamma}(T_f^*z)(t)-[\psi(t)-\psi(0)]^{1-\gamma}(T_f^*z^*)(t)\right|$$
$$\leq\frac{[\psi(t)-\psi(0)]^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}$$
$$\left|f(s,\tilde{\varphi}_s+\tilde{z}_s)-f\left(s,\tilde{\varphi}_s+\widetilde{z^*}_s\right)\right|ds$$
$$\leq\frac{[\psi(t)-\psi(0)]^{1-\gamma}}{\Gamma(\alpha)}M_f\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|\tilde{z}_s-\widetilde{z^*}_s\|_{\mathscr{B}}ds.$$

From **(H1)**, we obtain

$$\|\tilde{z}_s-\widetilde{z^*}_s\|_{\mathscr{B}}\leq K(t)\sup_{0\leq\tau\leq s}\left|\tilde{z}(\tau)-\widetilde{z^*}(\tau)\right|+M(t)\|\tilde{z}_0-\widetilde{z^*}_0\|_{\mathscr{B}}\tag{3.7}$$
$$\leq K_b\sup_{0\leq\tau\leq s}[\psi(\tau)-\psi(0)]^{1-\gamma}|z(\tau)-z^*(\tau)|$$
$$=K_b\left(\|z_0-z_0^*\|_{\mathscr{B}}+\|z-z^*\|_{C_{1-\gamma;\psi}}\right)$$
$$=K_b\|z-z^*\|_{\Omega^*_{\mathscr{B},\gamma,\psi}}.$$

It follows that

$$\left|[\psi(t)-\psi(0)]^{1-\gamma}\left(T_f^* z\right)(t)-[\psi(t)-\psi(0)]^{1-\gamma}\left(T_f^* z^*\right)(t)\right|$$
$$\leq M_f K_b \|z-z^*\|_{\Omega^*_{\mathscr{B},\gamma,\psi}} \frac{[\psi(t)-\psi(0)]^{1-\gamma}}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} ds$$
$$= \frac{(\psi(t)-\psi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} M_f K_b \|z-z^*\|_{\Omega^*_{\mathscr{B},\gamma,\psi}},$$

which implies

$$\|T_f^* z - T_f^* z^*\|_{\Omega^*_{\mathscr{B},\gamma,\psi}} \leq \frac{(\psi(b)-\psi(0))^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} M_f K_b \|z-z^*\|_{\Omega^*_{\mathscr{B},\gamma,\psi}}. \quad (3.8)$$

As the hypothesis $(\mathbb{F}_2)$, it follows that the operator T_f^* is a contraction mapping. An application of Banach fixed point theorem shows that T_f^* has a unique fixed point $z \in \Omega^*_{\mathscr{B},\gamma,\psi}$. Set $y = \widetilde{\varphi} + z$, then T_f has a unique fixed point $y \in \Omega_{\mathscr{B},\gamma,\psi}$ which is the unique solution to the ψ—Hilfer problem (1.2) in $\Omega_{\mathscr{B},\gamma,\psi}$. □

3.2. Ulam–Hyers–Mittag-Leffler (UHML) stability

In this section, we discuss the UHML stability of ψ—Hilfer problem (1.2). The following observations are taken from [15], [16], [25]. Let $y(t) = \widetilde{\varphi}(t) + z(t)$ is solution of (1.2) in $\Omega_{\mathscr{B},\gamma,\psi}$ with $y_t = \widetilde{\varphi}_t + \tilde{z}_t$ and let $x(t) = \widetilde{\varphi}(t) + w(t)$ satisfies of the inequality

$$\left|{}^H D_{0^+}^{\alpha,\beta,\psi} x(t) - f(t, x_t)\right| \leq \varepsilon E_\alpha(\psi(t)-\psi(0))^\alpha; \quad t \in I,$$

where $x_t = \widetilde{\varphi}_t + \widetilde{w}_t$.

Definition 3.3

The first equation of (1.2) is UHML stable with respect to $E_\alpha\left((\psi(t)-\psi(0))^\alpha\right)$ if there exists $C_{E_\alpha} > 0$ such that, for each $\varepsilon > 0$ and each solution $w \in \Omega^*_{\mathscr{B},\gamma,\psi}$ to the inequality (3.9), there exists a solution $z \in \Omega^*_{\mathscr{B},\gamma,\psi}$ to the problem (1.2) with

$$|z(t) - w(t)| \leq C_{E_\alpha} \varepsilon E_\alpha\left((\psi(t)-\psi(0))^\alpha\right); \qquad t \in I.$$

Remark 3.4

A function $w \in C_{1-\gamma;\psi}(I_1, \mathbb{R})$ is a solution of the inequality

$$\left|{}^H \mathscr{D}_{0^+}^{\alpha,\beta,\psi} w(t) - f(t, \widetilde{\varphi}_t + \widetilde{w}_t)\right| \leq \varepsilon E_\alpha(\psi(t)-\psi(0))^\alpha; \quad t \in I_2, \quad (3.9)$$

if and only if there exists a function $h \in C_{1-\gamma;\psi}(I_1, \mathbb{R})$ such that

(i)

$$|h(t)| \leq \varepsilon E_\alpha\left([\psi(t)-\psi(0)]^\alpha\right); \qquad t \in I_2;$$

(ii)

$${}^H \mathscr{D}_{0^+}^{\alpha,\beta,\psi} w(t) = f(t, \widetilde{\varphi}_t + \widetilde{w}_t) + h(t); \qquad t \in I_2.$$

Lemma 3.5

Let $\gamma = \alpha + \beta(1-\alpha)$ where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. If $w \in C_{1-\gamma;\psi}(I_1, \mathbb{R})$ satisfies the inequality (3.9), then w is a solution of the following integral inequality

$$\left|w(t) - \mathscr{A}_w - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(t, \widetilde{\varphi}_s + \widetilde{w}_s)\, ds\right|$$
$$\leq \varepsilon E_\alpha([\psi(t) - \psi(0)]^\alpha).$$

where

$$\mathscr{A}_w = \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} \varphi(0).$$

Proof

Indeed by Remark 3.4, we have that

$$^H\mathscr{D}_{0^+}^{\alpha,\beta,\psi} w(t) = f(t, \widetilde{\varphi}_t + \widetilde{w}_t) + h(t); \qquad t \in I_2.$$

It follows from Lemma 2.11 that

$$w(t) = \mathscr{A}_w + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(t, \widetilde{\varphi}_s + \widetilde{w}_s)\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s)\, ds.$$

From the last equality, Remark 3.4 and Lemma 2.10, we get

$$\left|w(t) - \mathscr{A}_w - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(t, \widetilde{\varphi}_s + \widetilde{w}_s)\, ds\right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |h(s)|\, ds$$
$$\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} E_\alpha([\psi(s) - \psi(0)]^\alpha)\, ds$$
$$\leq \varepsilon \mathscr{I}_{0^+}^{\alpha,\psi} E_\alpha([\psi(t) - \psi(0)]^\alpha)$$
$$= \varepsilon\left(E_\alpha\left[(\psi(t) - \psi(0))^\alpha\right] - 1\right)$$
$$\leq \varepsilon E_\alpha\left([\psi(t) - \psi(0)]^\alpha\right). \quad \square$$

Theorem 3.6

Assume that $(\mathbf{F}_1)$ and $(\mathbf{F}_2)$ are satisfied. If

$$\sup_{0 \leq s \leq b} [\psi(s) - \psi(0)]^{1-\gamma} < 1, \quad 0 \leq \gamma < 1. \tag{3.10}$$

Then the equation

$$^H\mathscr{D}_{0^+}^{\alpha,\beta,\psi} y(t) = f(t, y_t); \qquad t \in I_2, \tag{3.11}$$

is stable in the sense of UHML.

Proof

Let $\varepsilon > 0$, and let $w \in \Omega^*_{\mathscr{B},\gamma,\psi}$ be a function which satisfies the inequality

$$\left|{}^H\mathscr{D}^{\alpha,\beta,\psi}_{0^+} w(t) - f(t, \widetilde{\varphi}_t + \widetilde{w}_t)\right| \le \varepsilon E_\alpha\left([\psi(t) - \psi(0)]^\alpha\right); \qquad t \in I_2. \tag{3.12}$$

We denote by $z \in \Omega^*_{\mathscr{B},\gamma,\psi}$ the unique solution to the problem

$$\begin{cases} {}^H\mathscr{D}^{\alpha,\beta;\psi}_{0^+} z(t) = f(t, \widetilde{\varphi}_t + \widetilde{z}_t); \quad t \in I_2, \\ \mathscr{I}^{1-\gamma;\psi}_{0^+} z(0^+) = \mathscr{I}^{1-\gamma;\psi}_{0^+} w(0^+); \\ z(t) = w(t); \qquad t \in I^*. \end{cases} \tag{3.13}$$

According to Theorem3.2, we have

$$z(t) = \begin{cases} w(t); & t \in I^*, \\ \mathscr{A}_z + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \widetilde{\varphi}_s + \widetilde{z}_s)\, ds, & t \in I_2. \end{cases}$$

From the fact $\mathscr{A}_z = \mathscr{A}_w$ (since $I^{1-\gamma;\psi}_{0^+} z(0^+) = I^{1-\gamma;\psi}_{0^+} w(0^+)$ and $z(t) = w(t)$, we obtain

$$z(t) = \begin{cases} w(t); & t \in I^*, \\ \mathscr{A}_w + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \widetilde{\varphi}_s + \widetilde{z}_s)\, ds, & t \in I_2. \end{cases}$$

Thus, in the light of Lemma3.5, and ($\mathbf{F_1}$), we have for each$t \in I_2$

$$\begin{aligned} &|w(t) - z(t)| \\ &\le \left|w(t) - \mathscr{A}_w - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \widetilde{\varphi}_s + \widetilde{w}_s)\, ds\right| \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, \widetilde{\varphi}_s + \widetilde{w}_s) - f(s, \widetilde{\varphi}_s + \widetilde{z}_s)|\, ds \\ &\le \varepsilon E_\alpha\left([\psi(t) - \psi(0)]^\alpha\right) \\ &+ \frac{M_f}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|\widetilde{w}_s - \widetilde{z}_s\|_{\mathscr{B}} ds. \end{aligned} \tag{3.14}$$

In a similar way of the inequality (3.7), we obtain

$$\|\widetilde{w}_s - \widetilde{z}_s\|_{\mathscr{B}} \le K_b \sup_{0 \le \tau \le s} [\psi(\tau) - \psi(0)]^{1-\gamma} |w(\tau) - z(\tau)|. \tag{3.15}$$

From (3.14), (3.15) we have

$$|w(t) - z(t)| \le \varepsilon E_\alpha\left([\psi(t) - \psi(0)]^\alpha\right) + \frac{M_f}{\Gamma(\alpha)}$$
$$\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} K_b [\psi(s) - \psi(0)]^{1-\gamma} |w(s) - z(s)|\, ds.$$

Now, for all $u \in \Omega^*_{\mathscr{B},\gamma,\psi}$, we define the operator $T_1 : \Omega^*_{\mathscr{B},\gamma,\psi} \to \Omega^*_{\mathscr{B},\gamma,\psi}$ by

$$T_1 u(t) = \begin{cases} 0; & t \in I^*, \\ \varepsilon E_\alpha\left([\psi(t)-\psi(0)]^\alpha\right) + \frac{M_f K_b}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \\ [\psi(s)-\psi(0)]^{1-\gamma} u(s)\, ds, & t \in I_2. \end{cases}$$

We prove that T_1 is a Picard operator. For all $t \in I_2$ and for each $u, u^* \in C_{1-\gamma;\psi}(I_1, \mathbb{R})$, it follows from ($\mathbf{F_1}$) that

$$\begin{aligned} &\left|[\psi(t)-\psi(0)]^{1-\gamma}[(T_1 u)(t) - (T_1 u^*)(t)]\right| \\ &\leq \frac{M_f K_b [\psi(t)-\psi(0)]^{1-\gamma}}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[\psi(s)-\psi(0)]^{1-\gamma} |u(s) \\ &- u^*(s)|\, ds \\ &\leq \frac{M_f K_b [\psi(t)-\psi(0)]^{1-\gamma}}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \|u-u^*\|_{C_{1-\gamma,\psi}} ds \\ &= \frac{M_f K_b [\psi(t)-\psi(0)]^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} \|u-u^*\|_{C_{1-\gamma,\psi}}, \end{aligned}$$

which implies

$$\|T_1 u - T_1 u^*\|_{C_{1-\gamma,\psi}} \leq \frac{M_f K_b [\psi(b)-\psi(0)]^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} \|u-u^*\|_{C_{1-\gamma,\psi}}.$$

From ($\mathbf{F_2}$), the operator T_1 is a contraction mapping on $\Omega^*_{\mathscr{B},\gamma,\psi}$ via the norm $\|\cdot\|_{C_{1-\gamma;\psi}}$. Applying the Banach fixed point theorem to T_1, we see that T_1 is a Picard operator and $F_{T_1} = u^*$. Then we have for all $t \in I_2$

$$\begin{aligned} u^*(t) &= T_1 u^*(t) \\ &= \varepsilon E_\alpha\left([\psi(t)-\psi(0)]^\alpha\right) \\ &+ \frac{M_f K_b}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[\psi(s)-\psi(0)]^{1-\gamma} u^*(s)\, ds. \end{aligned}$$

Next, we prove that the solution u^* is increasing. Set $\sigma := \min_{s\in[0,b]} u^*(s) \in \mathbb{R}^+$, and for all $0 < t_1 < t_2 \leq b$, we have

$$
\begin{aligned}
&u^*(t_2) - u^*(t_1) \\
&= \varepsilon E_\alpha\left([\psi(t_2) - \psi(0)]^\alpha\right) - \varepsilon E_\alpha\left([\psi(t_1) - \psi(0)]^\alpha\right) \\
&+ M_f K_b \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s)\,(\psi(t_2) - \psi(s))^{\alpha-1}[\psi(s) - \psi(0)]^{1-\gamma} u^*(s)\,ds \\
&+ \frac{M_f K_b}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)\left[(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}\right] \\
&\times [\psi(s) - \psi(0)]^{1-\gamma} u^*(s)\,ds \\
&+ \frac{M_f K_b}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)\,(\psi(t_2) - \psi(s))^{\alpha-1}[\psi(s) - \psi(0)]^{1-\gamma} u^*(s)\,ds \\
&\geq \varepsilon E_\alpha\left([\psi(t_2) - \psi(0)]^\alpha\right) - \varepsilon E_\alpha\left([\psi(t_1) - \psi(0)]^\alpha\right) \\
&+ \left([\psi(t_2) - \psi(0)]^{1-\gamma} - [\psi(t_1) - \psi(0)]^{1-\gamma}\right) \frac{\sigma M_f K_b}{\Gamma(\alpha+1)} \\
&+ \left([\psi(t_2) - \psi(0)]^{\alpha} - [\psi(t_1) - \psi(0)]^{\alpha}\right) \frac{\sigma M_f K_b}{\Gamma(\alpha+1)} \\
&> 0.
\end{aligned}
$$

Therefore, u^* is increasing, and by (3.10), we get

$$
u^*(t) \leq \varepsilon E_\alpha[\psi(t) - \psi(0)]^\alpha + \frac{M_f K_b}{\Gamma(\alpha)} \int_0^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1} u^*(s)\,ds.
$$

Using Lemma2.18, for $t \in I_1$ we attain

$$
\begin{aligned}
&u^*(t) \leq \varepsilon E_\alpha[\psi(t) - \psi(0)]^\alpha E_\alpha(M_f K_b[\psi(t) - \psi(0)]^\alpha) \\
&\leq E_\alpha(M_f K_b[\psi(b) - \psi(0)]^\alpha)\varepsilon E_\alpha[\psi(t) - \psi(0)]^\alpha \\
&= C_{E_\alpha} \varepsilon E_\alpha\left([\psi(t) - \psi(0)]^\alpha\right),
\end{aligned}
$$

where $C_{E_\alpha} = E_\alpha(M_f K_b[\psi(b) - \psi(0)]^\alpha)$.

In particular, if $u = |w - z|$, then $u(t) \leq T_1 u(t)$ and by applying Lemma2.16, we obtain $u(t) \leq u^*(t)$. Therefore, it follows

$$
|x(t) - y(t)| = |w(t) - z(t)| \leq C_{E_\alpha} \varepsilon E_\alpha\left([\psi(t) - \psi(0)]^\alpha\right); \qquad t \in I.
$$

□

4. An example

Firstly, let ρ be a positive real constant and we define the functional space $\mathscr{B}_\rho$ by

$$
\mathscr{B}_\rho = \left\{ y \in C(I^*, \mathbb{R}) : \lim_{s \to -\infty} e^{\rho s} y(s) \text{ exist in } \mathbb{R} \right\},
$$

endowed with the following norm $\|y\|_\rho = \sup\{e^{\rho s}|y(s)| : -\infty < s \leq 0\}$.

Then $\mathscr{B}_\rho$ satisfies axioms **(H1)**, **(H2)** and **(H3)** with $K(t) = M(t) = 1$ and $\mathscr{H} = 1$. (see[31]).

Consider the ψ—Hilfer problem of FFDE

$$\begin{cases} {}^{H}\mathscr{D}_{0^+}^{\frac{1}{3},\frac{2}{3};2^t} y(t) = \frac{e^{-\rho t}\|y_t\|}{8+e^t}, & t \in (0,1], \\ \mathscr{I}_{0^+}^{\frac{5}{9};2^t} y(0^+) = 1, \\ y(t) = \varphi(t); & t \in (-\infty, 0], \end{cases} \tag{4.1}$$

where $\alpha = \frac{1}{3}, \beta = \frac{2}{3}, \gamma = \alpha + \beta - \alpha\beta = \frac{4}{9}$, $(0,b] = (0,1]$, $\psi(t) = 2^t$ for all $t \in (0,1]$, and $f(t,y) = \frac{e^{-\rho t} y}{8+e^t}$. Thus, for each $y, y^* \in \mathscr{B}_\rho$ and $t \in (0,1]$, we have

$$|f(t,x) - f(t,y)| = \left|\frac{e^{-\rho t}[x-y]}{8+e^t}\right| \le \frac{1}{8+e^t}\|x-y\|_\rho$$
$$\le \frac{1}{8}\|x-y\|_\rho.$$

Hence the conditions $(\mathbb{F}_1)$ holds with $M_f = \frac{1}{8}$. It can be checked that condition $(\mathbb{F}_2)$ is satisfied with $M_f = \frac{1}{8}$, $K_b = 1$, $\alpha = \frac{1}{3}$ and $\psi(t) = 2^t$ for $t \in (0,1]$ i.e.

$$\frac{M_f K_b}{\Gamma(\alpha+1)}[\psi(b) - \psi(0)]^\alpha = \frac{1}{8\Gamma\left(\frac{4}{3}\right)} \simeq 0.14 < 1.$$

Now all the hypotheses in Theorem3.2 are satisfied, so problem (4.1) has a unique solution on $(-\infty, 1]$.

Let us consider the following inequality

$$\left|{}^{H}\mathscr{D}_{0^+}^{\alpha,\beta,\psi} x(t) - \frac{e^{-\rho t} x_t}{8+e^t}\right| \le \varepsilon E_{\frac{1}{3}}\left(2^t - 1\right)^{\frac{1}{3}}; \quad t \in (-\infty, 1]$$

is satisfied. By applying Theorem3.6, the problem (4.1) is UHML stable with

$$|x(t) - y(t)| \le C_{E_{\frac{1}{3}}} \varepsilon E_{\frac{1}{3}}\left(\left(2^t - 1\right)^{\frac{1}{3}}\right); \qquad t \in (-\infty, 1],$$

where

$$C_{E_{\frac{1}{3}}} = E_{\frac{1}{3}}\left(\frac{1}{8}\right) = e^{\frac{3}{8}}\left(1 + 3\int_0^1 e^{-x^3}\left(\frac{1}{\Gamma\left(\frac{1}{3}\right)} + \frac{x}{\Gamma\left(\frac{2}{3}\right)}\right)dx\right) \approx 2.679472 e^{\frac{3}{8}}$$
$$> 0.$$

5. Conclusion

This paper studies a class of a nonlinear FFDEs with infinite delay involving ψ-Hilfer fractional derivative. The Picard operator method, Banach fixed point theorem and generalized Gronwall inequality are quite general and effective in our analysis, it is reasoned some adequate conditions for existence, uniqueness and UHLM stability of the solution to the considered FFDEs.

As a result, it is essential to develop the concepts of stability for the proposed problem. The advantages of the problem considered and the importance of obtained results have been provided in the introduction section. It is the first work concerning FFDEs with infinite delay involving ψ-

Hilfer fractional derivative. We trust the reported results here will have a positive impact on the development of further applications in engineering and applied sciences.

CRediT authorship contribution statement

Mohammed S. Abdo: Data curation, Investigation, Writing - original draft. **Satish K. Panchal:** Supervision, Validation. **Hanan A. Wahash:** Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The author would like to thank reviewers and editor for useful discussions and helpful comments that improved the manuscript.

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