Some Ostrowski Type Inequalities for Double Integrals on Time Scales

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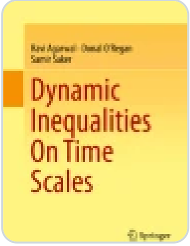
Abstract

The main objective of this paper is to study some Ostrowski type inequalities for double integrals on Time Scales. Some other interesting inequalities are also given.



1 Introduction

In year 1988 the German mathematician Stefan Hilger in his Ph.D. dissertation has initiated the study of time scales calculus which unifies the theory of both differential and difference calculus [9]. Dynamical equations and inequality's can be used studying various properties and model many phenomena in economics [5], biological systems [24] and various systems in neural network [12].

In [8] Bohner and Matthews have given the Ostrowski inequality and Montgomery identity on time scales which are helpful in obtaining various results. Some results on Ostrowski and Gruss inequality were obtained by N. Ahmad, W. Liu and others [2, 13, 23]. Recently in [4, 11, 14, 16, 21, 25] authors have obtained some new Ostrowski type inequalities. Weighted Ostrowski and Trapezoid inequalities on time scales were studied by W. Liu and others in [17–20]. In [22] M. Sarikya have studied some weighted Ostrowski and Chebyshev type inequalities on time scales. Motivated by the results in the above paper we obtain some Ostrowski and Trapezoid type inequalities for double integrals on time scales.

In what follows the time scale \(\mathbb{T}\) is a nonempty closed subset of $\mathbb{R}$. Let \(t \in\mathbb{T}\) the mapping \(\sigma,\rho: \mathbb{T} \to\mathbb{T}\) are defined as $\sigma (t) = \inf \{ {s \in\mathbb{T}:s > t} \}$ and $\rho (t) = \sup \{ {s \in\mathbb{T}:s < t} \}$ and are called the forward and backward jump operators respectively.

We say that $f:\mathbb{T} \to\mathbb{R}$ is rd-continuous provided f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left sided limit at each left dense point of $\mathbb{T}$. Now we denote C_{rd} as the set of rd-continuous function defined on $\mathbb{T}$. Let $\mathbb{T}_{1}$ and $\mathbb{T}_{2}$ be two time scales with at least two points and consider the time scales intervals $\overline{\mathbb{T}}_{1} = [{x_{0} ,\infty}) \cap \mathbb{T}_{1}$ and $\overline{\mathbb{T}}_{2} = [{y_{0} , \infty}) \cap\mathbb{T}_{2}$ for $x_{0} \in\mathbb{T}_{1}$ and $y_{0} \in\mathbb{T}_{2}$ and $\varOmega=\mathbb{T}_{1} \times \mathbb{T}_{2}$. Let σ_{1}, ρ_{1}, Δ_{1} and σ_{2}, ρ_{2}, Δ_{2} denote the forward jump operators, backward jump operators and the delta differentiation operator respectively on $\mathbb{T}_{1}$ and $\mathbb{T}_{2}$. Let $a< b$ be points in $\mathbb{T}_{1}$, $c< d$ are point in $\mathbb{T}_{2}$, $[a,b)$ is the half closed bounded interval in $\mathbb{T}_{1}$, and $[c,d)$ is the half closed bounded interval in $\mathbb{T}_{2}$.

We say that a real valued function f on $\mathbb{T}_{1}\times \mathbb{T}_{2}$ at $(t_{1},t_{2}) \in\overline{\mathbb{T}}_{1} \times\overline{\mathbb{T}}_{2}$ has a Δ_{1} partial derivative $f^{\Delta_{1}}(t_{1},t_{2})$ with respect to t_{1} if for each $\epsilon>0$ there exists a neighborhood $U_{t_{1} }$ of t_{1} such that

$$ \bigl| {f \bigl({\sigma_{1} ({t_{1} }) ,t_{2} } \bigr) - f ({s,t_{2} }) - f^{\Delta_{1} } ({t_{1} ,t_{2} }) \bigl({\sigma_{1} ({t_{1} }) - s} \bigr) } \bigr| \le\varepsilon \bigl| {\sigma_{1} ({t_{1} }) - s} \bigr| , $$

for each $s \in U_{t_{1}}$, $t_{2} \in\mathbb{T}_{2}$. We say that f on $\mathbb{T}_{1}\times\mathbb{T}_{2}$ at $(t_{1},t_{2}) \in \overline{\mathbb{T}}_{1} \times\overline{\mathbb{T}}_{2}$ has a Δ_{2} partial derivative $f^{\Delta_{2}}(t_{1},t_{2})$ with respect to t_{2} if for each $\eta>0$ there exists a neighborhood $U_{t_{2} }$ of t_{2} such that

$$ \bigl| {f \bigl({t_{1} ,\sigma_{2} ({t_{2} }) } \bigr) - f ({t_{1} ,l}) - f^{\Delta_{2} } ({t_{1} ,t_{2} }) \bigl({\sigma_{2} ({t_{2} }) - l} \bigr) }\bigr| \le\eta \bigl| {\sigma_{2} ({t_{2} }) - l}\bigr| , $$

for all $l \in U_{t_{2}}$, $t_{1} \in\mathbb{T}_{1}$. The function f is called rd-continuous in t_{2} if for every $\alpha_{1} \in\mathbb{T}_{1}$, the function $f(\alpha_{1},.)$ is rd-continuous on $\mathbb{T}_{2}$. The function f is called rd-continuous in t_{1} if for every $\alpha_{2} \in\mathbb{T}_{2}$ the function $f(.,\alpha_{2})$ is rd-continuous on $\mathbb{T}_{1}$.

Let CC_{rd} denote the set of functions $f(t_{1},t_{2})$ on $\mathbb{T}_{1} \times\mathbb{T}_{2}$ where f is rd continuous in t_{1} and t_{2}. Let CC'_{rd} denotes the set of all functions CC_{rd} for which both the Δ_{1} partial derivative and Δ_{2} partial derivative exists and are in CC_{rd}.

The basic information on time scales and inequalities can be found in [1, 3, 6, 7].

2 Ostrowski Inequalities for Double Integrals on Time Scales

Now we give Ostrowski Inequalities for double integrals on time scales

Theorem 1

Let $f,g \in CC'_{rd} ({ [{a,b}] \times [{c,d}] ,\mathbb{R}})$ *and* $f^{\Delta_{2} \Delta_{1} } ({x,y})$, $g^{\Delta_{2} \Delta_{1} } ({x,y})$ *exist rd-continuous on* $[a,b] \times[c,d]$. *Then*

$$\begin{aligned} &\Biggl|\int\limits_{a}^{b} \int\limits_{c}^{d} \biggl\{ f ({x,y}) g ({x,y}) - \frac{1}{2} \bigl[P \bigl({f ({x,y}) } \bigr) g ({x,y}) + P \bigl(g ({x,y}) \bigr) f ({x,y}) \bigr] \biggr\} {\Delta_{2} y \Delta_{1} x} \Biggr| \\ &\quad \le\frac{1}{8} ({b - a}) ({d - c}) \int\limits_{a}^{b} {\int\limits_{c}^{d} { \Bigl[\bigl| {g ({x,y}) }\bigr| } } \bigl\| {f^{\Delta_{2} \Delta_{1} } } \bigr\| _{\infty} + f ({x,y}) \bigl\| g^{\Delta_{2} \Delta _{1} } \bigr\| \Bigr] _{\infty}\Delta_{2} y\Delta_{1} x, \end{aligned}$$

(2.1)

where

$$\begin{aligned} P \bigl({f ({x,y}) } \bigr) =& \frac{1}{2} \bigl[{f \bigl({\sigma_{1} (s) ,y} \bigr) + f \bigl({x,\sigma_{2} (t) } \bigr) + f ({x,d}) + f ({b,y}) } \bigr] \\ &{}- \frac{1}{4} \bigl[{f \bigl({\sigma_{1} (s) , \sigma_{2} (t) } \bigr) + f \bigl({\sigma_{1} (s) ,d} \bigr) + f \bigl({b,\sigma_{2} (t) } \bigr) + f ({b,d}) } \bigr] , \end{aligned}$$

(2.2)

and

$$\begin{aligned} &Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) \\ &\quad = \int\limits_{\sigma_{1} (s) }^{b} {\int \limits_{\sigma_{2} (t) }^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta- \int \limits_{\sigma_{1} (s) }^{b} {\int\limits_{y}^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta \\ &\quad \quad {} - \int\limits_{x}^{b} {\int\limits_{\sigma_{2} (t) }^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta _{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta+ \int\limits_{x}^{b} {\int\limits_{y}^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta. \end{aligned}$$

(2.3)

Similarly $P(g(x,y))$ *and* $Q ({g^{\Delta_{2} \Delta_{1} } ({x,y}) })$ *are defined similar to* (2.2) *and* (2.3).

Proof

From the hypotheses we have for $(x,y) \in[a,b] \times[c,d]$

$$\begin{aligned} &\int\limits_{\sigma_{1} (s) }^{b} {\int \limits_{\sigma_{2} (t) }^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta \\ &\quad = \int\limits_{\sigma_{1} (s) }^{x} { \biggl[{{\frac{{\partial f ({\eta,\tau}) }}{{\Delta_{1} \eta}}} \biggl| _{\sigma_{2} (t) }^{y} } \biggr] } \Delta_{1} \eta \\ &\quad = \int\limits_{\sigma_{1} (s) }^{x} { \biggl[{\frac{

{\partial f ({\eta,y}) }}{{\Delta_{1} \eta}} - \frac{ {\partial f ({\eta,\sigma_{2} (t) }) }} { {\Delta_{1} \eta}}} \biggr] } \Delta_{1} \eta \\ &\quad = {f ({\eta,y}) } | _{\sigma_{1} (s) }^{x} - {f \bigl({\eta,\sigma_{2} (t) } \bigr) } \bigl| _{\sigma_{1} (s) } ^{x} \\ &\quad = f ({x,y}) - f \bigl({\sigma_{1} (s) ,y} \bigr) - f \bigl({x,\sigma_{2} (t) } \bigr) + f \bigl({\sigma_{1} (s) ,\sigma_{2} (t) } \bigr) . \end{aligned}$$

(2.4)

Similarly we have

$$\begin{aligned} &\int\limits_{\sigma_{1} (s) }^{b} {\int\limits_{y}^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta \\ &\quad = - f ({x,y}) - f \bigl({\sigma_{1} (s),d} \bigr) + f ({x,d}) + f \bigl({\sigma_{1} (s) ,y} \bigr) , \end{aligned}$$

(2.5)

$$\begin{aligned} &\int\limits_{x}^{b} {\int\limits_{\sigma_{2} (t) }^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta \\ &\quad = - f ({x,y}) - f \bigl({b,\sigma_{2} (t) } \bigr) + f \bigl({x,\sigma_{2} (t) } \bigr) + f \bigl({b,\sigma_{2} (t) } \bigr) , \end{aligned}$$

(2.6)

and

$$\begin{aligned} \int\limits_{x}^{b} {\int\limits_{y}^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta = f ({x,y}) + f ({b,d}) - f ({x,d}) - f ({b,y}) . \end{aligned}$$

(2.7)

Adding above identities we have

$$\begin{aligned} & 4f ({x,y}) - 2f \bigl({\sigma_{1} (s) ,y} \bigr) - 2f \bigl({x,\sigma_{2} (t) } \bigr) - 2f ({x,d}) - 2f ({b,y}) \\ &\quad \quad {}+ f \bigl(

{\sigma_{1} (s) ,\sigma_{2} (t) } \bigr) + f \bigl({\sigma_{1} (s) ,d} \bigr) + f \bigl({b,\sigma_{2} (t) } \bigr) + f ({b,d}) \\ &\quad = \int\limits_{\sigma_{1} (s) }^{b} {\int \limits_{\sigma_{2} (t) }^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta- \int \limits_{\sigma_{1} (s) }^{b} {\int\limits_{y}^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta \\ &\quad \quad {}- \int\limits_{x}^{b} {\int\limits_{\sigma_{2} (t) } ^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta _{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta+ \int\limits_{x}^{b} {\int\limits_{y}^{d} {\frac{{\partial^{2} f ({\eta,\tau}) }}{{\Delta_{2} \tau\Delta_{1} \eta}}} } \Delta_{2} \tau\Delta_{1} \eta. \end{aligned}$$

(2.8)

From (2.3), (2.4) and (2.8) we have

$$\begin{aligned} f ({x,y}) - P \bigl({f ({x,y}) } \bigr) = \frac{1}{4}Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y})}\bigr) , \end{aligned}$$

(2.9)

for \(x,y \in[a,b] \times[c,d]\).

Similarly for function \(g\) we have

$$\begin{aligned} g ({x,y}) - P \bigl({g ({x,y}) } \bigr) = \frac{1}{4}Q \bigl({g^{\Delta_{2} \Delta_{1} } ({x,y})} \bigr) , \end{aligned}$$

(2.10)

for \(x,y \in[a,b] \times[c,d]\). Multiplying (2.9) and (2.10) by \(g(x,y)\) and \(f(x,y)\) and adding the resulting identities we get

$$\begin{aligned} &2f ({x,y}) g ({x,y}) - g ({x,y}) P \bigl({f ({x,y}) } \bigr) - f(x,y)P \bigl({g ({x,y}) } \bigr) \\ &\quad = \frac{1}{4}g ({x,y}) Q \bigl({f^{\Delta_{2} \Delta _{1} } ({x,y}) } \bigr) + \frac{1}{4}f (x,y) Q \bigl({g^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr). \end{aligned}$$

(2.11)

Integrating (2.11) over $[a,b] \times[c,d]$ we have

$$\begin{aligned} &\int\limits_{a}^{b}\int\limits_{c}^{d} \biggl[f(x,y) g(x,y)- \frac{1}{2} \bigl[P \bigl(f (x,y) \bigr) g (x,y) + P \bigl(g (x,y) \bigr) f (x,y) \bigr] \biggr] \Delta_{2} y\Delta_{1} x \\ &\quad = \frac{1}{8} \int\limits_{a}^{b} \int\limits_{c}^{d} \bigl[Q \bigl(f^{\Delta_{2} \Delta_{1}} (x,y) \bigr)g (x,y) + Q \bigl(g^{\Delta_{2} \Delta_{1}} (x,y) \bigr) f (x,y) \bigr]\Delta_{2} y\Delta_{1} x. \end{aligned}$$

(2.12)

From the properties of modulus we have

$$\begin{aligned} \bigl| {Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) }\bigr| & \le\int\limits_{a}^{b} {\int\limits_{c}^{d}{\bigl| {f^{\Delta_{2} \Delta_{1} } ({t,s}) }\bigr| } } \Delta_{2} s\Delta_{1} t \\ & \le \bigl\| {f^{\Delta_{2} \Delta_{1} } } \bigr\| _{\infty} ({b - a}) ({d - c}) , \end{aligned}$$

(2.13)

and

$$\begin{aligned} \bigl| {Q \bigl({g^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) } \bigr| &\le\int\limits_{a}^{b} {\int\limits_{c}^{d} {\bigl| {g^{\Delta_{2} \Delta_{1} } ({t,s}) }\bigr| } } \Delta_{2} s\Delta_{1} t \\ &\le \bigl\| {g^{\Delta_{2} \Delta_{1} } }\bigr\| _{\infty} ({b - a}) ({d - c}) . \end{aligned}$$

(2.14)

From (2.12), (2.13) and (2.14) we have

$$\begin{aligned} &\Biggl| \int\limits_{a}^{b} \int\limits_{c}^{d} \Biggl[f (x,y)g (x,y)- \frac{1}{2}\biggl[P \bigl(f (x,y) \bigr)g (x,y) + P \bigl(g (x,y) \bigr) f (x,y)\biggr]\Biggr]\Delta_{2} y\Delta_{1} x \Biggr| \\ &\quad \le\frac{1}{8}\int\limits_{a}^{b} {\int\limits_{c}^{d} { \Bigl[{ \bigl| {g ({x,y}) } \bigr| } } } \bigl|

{Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr)} \bigr| + \bigl| {f ({x,y}) } \bigr| \bigl|{Q \bigl({g^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr)} \bigr| \Bigr] \Delta_{2} y\Delta_{1} x \\ &\quad \le\frac{1}{8}\int\limits_{a}^{b} {\int\limits_{c}^{d} { \Biggl[{ \bigl| {g ({x,y}) } \bigr| } } } \int\limits_{a}^{b} {\int\limits_{c}^{d} { \bigl| {f^{\Delta_{2} \Delta_{1} } ({t,s}) } \bigr| } } \Delta_{2} s\Delta_{1} t \\ &\quad \quad {}+ \bigl| {f ({x,y}) } \bigr| \int\limits_{a}^{b} {\int\limits_{c}^{d} { \bigl| {g^{\Delta_{2} \Delta_{1} } ({t,s}) }\bigr| } } \Delta_{2} s\Delta_{1} t\Biggr] \Delta_{2} y\Delta_{1} x \\ &\quad \le\frac{1}{8} ({b – a}) ({d – c}) \int\limits_{a}^{b} {\int\limits_{c}^{d} { \Bigl[{ \bigl| {g ({x,y}) } \bigr| } } } \bigl\| {f^{\Delta_{2}\Delta_{1} } } \bigr\| _{\infty} + \bigl|{f ({x,y}) }\bigr| \bigl\| {g^{\Delta_{2} \Delta_{1} } } \bigr\| _{\infty} \Bigr] \Delta_{2} y\Delta_{1} x, \end{aligned}$$

(2.15)

which is required inequality. □

Now we give the continuous and Discrete equivalent version of above inequality where \(\mathbb{T}=\mathbb{R}\) and \(\mathbb{T}=\mathbb{Z}\).

Corollary 1

(Continuous Case)

If we put \(\mathbb{T}_{1}=\mathbb{T}_{2}=\mathbb{R}\) *we have*

$$\begin{aligned} &\Biggl| {\int\limits_{a}^{b} {\int\limits_{c}^{d} { \biggl[{f ({x,y}) } } } } g ({x,y}) { { – \frac{1}{2} \bigl[{P \bigl({f ({x,y}) } \bigr) g ({x,y}) + P \bigl({g ({x,y}) } \bigr) f ({x,y}) } \bigr] } \biggr] dydx}\Biggr| \\ &\quad \le\frac{1}{8} ({b – a}) ({d – c})\int\limits_{a}^{b} {\int\limits_{c}^{d} { \Bigl[{ \bigl| {g ({x,y}) }\bigr| } } } \| {D_{2} D_{1} f} \| _{\infty} + \bigl| {f ({x,y}) } \bigr| \|{D_{2} D_{1} g} \| _{\infty} \Bigr] dydx, \\ &P\bigl(f(x,y)\bigr) \\ &\quad = \frac{1}{2} \bigl[{f ({x,c}) + f ({x,d}) + f ({a,y}) + f ({b,y}) } \bigr] \\ &\qquad {}– \frac{1}{4} \bigl[{f ({a,c}) + f ({a,d}) + f ({b,c}) + f ({b,d}) } \bigr], \\ &Q \bigl({D_{2} D_{1} f ({x,y}) } \bigr) \\ &\quad = \int\limits_{a}^{x} {\int\limits_{c}^{y} {D_{2} D_{1} f ({t,s}) } } dsdt – \int\limits_{a}^{x} {\int\limits_{y}^{d} {D_{2} D _{1} f ({t,s}) } } dsdt \\ &\quad \quad

{}- \int\limits_{x}^{b} {\int\limits_{c}^{y} {D_{2} D_{1} f ({t,s}) } } dsdt + \int\limits_{x}^{b} {\int\limits_{y}^{d} {D_{2} D_{1} f ({t,s}) } } dsdt . \end{aligned}$$

Similarly\(P(g(x,y))\)*and*\(Q(D_{2} D_{1} g(x,y))\)*are defined which is Ostrowski inequality for Double integral*.

Corollary 2

(Discrete Case)

If\(\mathbb{T}_{1}=\mathbb{T}_{2}=\mathbb{Z}\)*and*\(a=c=0\), \(b=k \in \mathbb{N}\)*and*\(d=r \in\mathbb{N}\). *Then*

$$\begin{aligned} & \Biggl|{\sum\limits_{x = 1}^{k} {\sum\limits_{y = 1}^{r} { \biggl\{ {f ({x,y}) g ({x,y}) } } } } - \frac{1}{2} \bigl({g ({x,y}) P \bigl({f ({x,y}) } \bigr) + f ({x,y}) P \bigl({g ({x,y}) } \bigr) } \bigr) \biggr\} \Biggr| \\ &\quad \le\frac{1}{8}kr\sum\limits_{x = 1}^{k} {\sum\limits_{y = 1}^{r} { \Bigl[{ \bigl| {g ({x,y}) } \bigr| \| {\Delta _{2} \Delta_{1} f} \| _{\infty}+ \bigl| {f ({x,y}) } \bigr| \| {\Delta_{2} \Delta_{1} g} \| _{\infty}} \Bigr] } }, \\ &P \bigl({f ({x,y}) } \bigr) \\ &\quad = \frac{1}{2} \bigl[{f ({x,1}) + f ({x,r + 1}) + f(1,y) + f(k + 1,y)} \bigr] \\ &\quad \quad {}- \frac{1}{4} \bigl[{f ({1,1}) + f ({1,r + 1}) + f(k + 1,1) + f(k + 1,r + 1)} \bigr] , \\ &Q \bigl({\Delta_{2} \Delta_{1} f ({x,y}) } \bigr)\sum\limits_{s = 1}^{x - 1} {\sum\limits_{t = 1}^{y - 1} {\Delta_{2} \Delta_{1} f ({s,t}) } } - \sum\limits_{s = 1}^{x - 1} {\sum\limits_{t = 1}^{m} {\Delta_{2} \Delta_{1} f ({s,t})}} \\ &\quad \quad {}- \sum\limits_{s = x}^{k} {\sum\limits_{t = 1}^{y - 1} {\Delta_{2} \Delta_{1} f ({s,t}) } } + \sum\limits_{s = x}^{k} {\sum\limits_{t = y}^{r} {\Delta_{2} \Delta_{1} f ({s,t})}}, \end{aligned}$$

which is Discrete Ostrowski Inequality.

In [15] authors have obtained the Ostrowski type inequality for double integral for single function \(f\) but in this paper we have obtained the Ostrowski inequality for two variables \(f\) and \(g\) on double integral on time scales.

Now we give some Ostrowski inequality for double integrals.

Theorem 2

Let f, g, $P(f(x,y))$, $P(g(x,y))$, $f^{\Delta_{2} \Delta_{1}}$, $g^{\Delta_{2}\Delta_{1}}$ *be as in Theorem* 2.1 *then*

$$\begin{aligned} &\Biggl| \int\limits_{a}^{b} \int\limits_{c}^{d} \Bigl\{ f(x,y) g(x,y)- \bigl[P\bigl(f(x,y)\bigr)g(x,y) + P\bigl(g(x,y)\bigr) f(x,y) \\ &\quad \quad {}- P\bigl(f(x,y)\bigr) P\bigl(g(x,y)\bigr)\bigr]\Bigr\} \Delta_{2} y\Delta_{1} x \Biggr| \\ &\quad \le\frac{1}{{16}} \bigl\{ { ({b - a}) ({d - c}) } \bigr\} ^{2} \bigl\| {f^{\Delta_{2} \Delta_{1} } } \bigr\| _{\infty} \bigl\| {g^{\Delta_{2} \Delta_{1} } }\bigr\| _{\infty}, \end{aligned}$$

(2.16)

for $[x,y] \in[a,b] \times[c,d]$.

Proof

From (2.9) and (2.10) we have

$$\begin{aligned} f(x,y) - P \bigl({f ({x,y}) } \bigr) = \frac{1}{4}Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr), \end{aligned}$$

(2.17)

and

$$\begin{aligned} g(x,y) - P \bigl({g ({x,y}) } \bigr) = \frac{1}{4}Q \bigl({g^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr), \end{aligned}$$

(2.18)

for $[x,y] \in[a,b] \times[c,d]$. □

Multiplying left hand side and right hand side of (2.17) and (2.18) we get

$$\begin{aligned} &f(x,y)g(x,y) - \bigl[{f(x,y)P \bigl({g ({x,y}) } \bigr) + g(x,y)P \bigl({f ({x,y}) } \bigr) } \bigr] \\ &\quad = \frac{1}{{16}}Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) Q \bigl({g^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) . \end{aligned}$$

(2.19)

Integrating (2.19) over $[a,b] \times[c,d]$ and from the properties of modulus we have

$$\begin{aligned} & \Biggl| {\int\limits_{a}^{b} {\int\limits_{c}^{d} { \Bigl\{ {f ({x,y}) g ({x,y}) } } } } - \bigl[P \bigl({f ({x,y}) } \bigr) g(x,y) + P \bigl(g (x,y)\bigr)f (x,y) \\ &\qquad{} { { - P \bigl({f ({x,y}) } \bigr) P \bigl({g ({x,y}) } \bigr) } \bigr] \Bigr\} \Delta_{2} y\Delta_{1} x} \Biggr| \\ &\quad \le\frac{1}{{16}}\int\limits_{a}^{b} {\int\limits_{c}^{d} { \bigl|{Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr)}\bigr| } } \bigl|{Q \bigl({g^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) }\bigr| \Delta_{2} y\Delta_{1} x. \end{aligned}$$

(2.20)

Now using (2.13) and (2.14) in (2.15) we get the required inequality (2.16).

Now we give continuous and discrete version of the inequality (2.16) where $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$ which is as follows

Corollary 3

(Continuous Case)

If we put $\mathbb{T}_{1}=\mathbb{T}_{2}=\mathbb{R}$ *in above we get*

$$\begin{aligned} & \Biggl| {\int\limits_{a}^{b} {\int\limits_{c}^{d} { \Bigl\{ {f ({x,y}) g ({x,y}) } } } } - \bigl[P \bigl({f ({x,y}) } \bigr) g ({x,y}) + P \bigl({g ({x,y}) } \bigr) f ({x,y}) \\ &\qquad{} { { - P \bigl({f ({x,y}) } \bigr) P \bigl({g ({x,y}) } \bigr) } \bigr] \Bigr\} dydx} \Biggr| \\ &\quad \le\frac{1}{{16}}\int\limits_{a}^{b} {\int\limits_{c}^{d} { \| {D_{2} D_{1} f} \| _{\infty} \| {D_{2} D_{1} g} \| _{\infty}} } dydx, \end{aligned}$$

where f, g, P, Q, $D_{2} D_{1} f$, $D_{2} D_{1} g$ *is as in Corollary* 2.1. *Which is Ostrowski type inequality for double integral.*

Corollary 4

(Discrete Case)

If $\mathbb{T}_{1}=\mathbb{T}_{2}=\mathbb{Z}$ *and* $a=c=0$, $b=k \in N$ *and* $d=r \in N$. *Then*

$$\begin{aligned} & \Biggl| \sum\limits_{x = 1}^{k} \sum\limits_{y = 1}^{r} \Bigl\{ f (x,y) g (x,y) - \bigl[g(x,y) P\bigl(f(x,y)\bigr) + f(x,y)P\bigl(g(x,y)\bigr) \\ &\quad \quad {}- P\bigl(f(x,y)\bigr)P\bigl(g(x,y)\bigr)\bigr]\Bigr\} \Biggr| \\ &\quad \le\frac{1}{{16}} ({kr}) ^{2} \int\limits_{a}^{b}{\int\limits_{c}^{d} { \| {\Delta_{2} \Delta_{1} f} \|_{\infty} \| {\Delta_{2} \Delta_{1} g} \| _{\infty}} }, \end{aligned}$$

where P, Q *are as in Corollary* 2.2, *which is discrete Ostrowski type inequality*.

In [8, 10] authors have obtained the results on Ostrowski inequality for three functions of single variables. Here we have obtained more interesting and generalized results for Ostrowski inequality having two functions of two variables on time scales. In [19] authors have studied the weighted Ostrowski and Ostrowski-Gruss type inequality on time scales for two functions of single variables. The obtained inequalities is more generalization Ostrowski Inequality on two variable for double integrals on time scale calculus.

3 Trapezoid Type Inequality on Time Scales

In [17] authors have obtained weighted Trapezoid type inequality on time scales for a function of single variables. Motivated by the results in above paper here we obtain the Trapezoid type inequality on time scales for functions of two variables.

Now we give the dynamic Trapezoid type inequality on time scales of following type.

Theorem 1

Let f, $f^{\Delta_{2} \Delta_{1} }$ *be as in Theorem* 2.1. *Then*

$$\begin{aligned} & \Biggl| \int\limits_{a}^{b} {\int\limits_{c}^{d} {f ({t,s}) \Delta_{2} s\Delta_{1} } } t - \frac{1}{2} \Biggl[{ ({d - c}) \int\limits_{a}^{b} {f \bigl({t,\sigma_{2} (t) } \bigr) + f ({t,d}) } } \Biggr] \\ &\quad \quad {} + ({b - a}) \int\limits_{c}^{d} { \bigl[{f \bigl({\sigma_{1} (s) ,s} \bigr) + f ({b,s}) } \bigr] } \Delta_{2} s \Biggr] \\ &\quad \quad {}+ \frac{1}{4} ({b - a}) ({d - c}) \bigl[{f \bigl({\sigma_{1} (s) ,\sigma_{2} (t) } \bigr) + f \bigl({\sigma_{1} (s) ,d} \bigr) } + f \bigl(b,\sigma_{2} (t)\bigr) + f(b,d)\bigr]\Biggr| \\ &\quad = \frac{1}{4}\int\limits_{a}^{b}\int\limits_{c}^{d}\bigl|{f^{\Delta_{2}\Delta_{1}}({t,s})}\bigr|\Delta_{2} s\Delta_{2} t. \end{aligned}$$

(3.1)

Proof

From the proof of Theorem 2.1 we have

$$\begin{aligned} f ({x,y}) &= \frac{1}{2} \bigl[{f \bigl({\sigma_{1} (s) ,y} \bigr) + f \bigl({x,\sigma_{2} (t) } \bigr) + f ({x,d}) + f ({b,y}) } \bigr] \\ &\quad {}+ \frac{1}{4} \bigl[{f \bigl({\sigma_{1} (s) , \sigma_{2} (t) } \bigr) + f \bigl({\sigma_{1} (s) ,d} \bigr) + f \bigl({b,\sigma_{2} (t) } \bigr) + f ({b,d}) } \bigr] \\ &= \frac{1}{4}Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) , \end{aligned}$$

(3.2)

for $[x,y] \in[a,b] \times[c,d]$.

Integrating (3.2) over $[a,b] \times[c,d]$ we get

$$\begin{aligned} &\int\limits_{a}^{b} {\int\limits_{c}^{d} {f ({t,s})} } \Delta_{2} s\Delta_{1} t - \frac{1}{2} \Biggl[{ ({d - c}) \int\limits_{a}^{b} { \bigl[{f \bigl({t,\sigma_{2} (t) } \bigr) + f ({t,d}) } \bigr] } } \Delta_{1} t \\ &\quad \quad {} + { ({b - a}) \int\limits_{c}^{d} { \bigl[{f \bigl({\sigma_{1} (s) ,s} \bigr) + f ({b,s}) } \bigr] \Delta_{2} s} } \Biggr] \\ &\quad \quad {} + \frac{1}{4} ({b - a}) ({d - c}) \bigl[{f \bigl({\sigma_{1} (s) ,\sigma_{2} (t) } \bigr) + f \bigl({\sigma_{1} (s) ,d} \bigr) } + f \bigl({b,\sigma_{2} (t) } \bigr) + f ({b,d}) \bigr] \\ &\quad = \frac{1}{4}\int\limits_{a}^{b} {\int\limits_{c}^{d} {Q \bigl({f^{\Delta_{2} \Delta_{1} } ({t,s}) } \bigr) } } \Delta_{2} s\Delta_{1} t. \end{aligned}$$

(3.3)

From the property of modulus and integrals we have

$$\begin{aligned} \bigl| {Q \bigl({f^{\Delta_{2} \Delta_{1} } ({x,y}) } \bigr) } \bigr| \le\int\limits_{a}^{b} {\int\limits_{c}^{d} { \bigl| {f^{\Delta_{2} \Delta_{1} } ({t,s}) }\bigr| } } \Delta_{2} s\Delta_{1} t. \end{aligned}$$

(3.4)

From (3.3) and (3.4) we have

$$\begin{aligned} & \Biggl| {\int\limits_{a}^{b} {\int\limits_{c}^{d} {f ({t,s}) } } \Delta_{2} s\Delta_{1} t} - \frac{1}{2} \Biggl[{ ({d - c}) \int\limits_{a}^{b} { \bigl[{f \bigl({t,\sigma_{2} (t) } \bigr) + f ({t,d}) } \bigr] } } \Delta_{1} t \\ &\quad \quad {} + { ({b - a}) \int\limits_{c}^{d} { \bigl[{f \bigl({\sigma_{1} (s) ,s} \bigr) + f ({b,s}) } \bigr] \Delta_{2} s} } \Biggr] \\ &\quad \quad {}+ \frac{1}{4} ({b - a}) ({d - c}) \bigl[{f \bigl({\sigma_{1} (s) ,\sigma_{2} (t) } \bigr) + f \bigl({\sigma_{1} (s) ,d} \bigr) } { { + f \bigl({b,\sigma_{2} (t) } \bigr) + f ({b,d}) } } \bigr] \Biggr| \\ &\quad \le\frac{1}{4}\int\limits_{a}^{b} {\int\limits_{c}^{d} { \bigl|{Q \bigl({f^{\Delta_{2} \Delta_{1} } ({t,s}) } \bigr) } \bigr| } } \Delta_{2} s\Delta_{1} t \\ &\quad \le\frac{1}{4} ({b - a}) ({d - c}) \int\limits_{a}^{b} {\int\limits_{c}^{d} { \bigl|{f^{\Delta_{2}\Delta_{1} } ({t,s}) }\bigr| } } \Delta_{2} s\Delta _{1} t, \end{aligned}$$

(3.5)

which is required inequality. Now we give the Continuous and discrete version of Trapezoid inequality when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$. □

Corollary 1

(Continuous Case)

If we put $\mathbb{T}_{1}=\mathbb{T}_{2}=\mathbb{R}$ *in above we get*

$$\begin{aligned} & \int\limits_{a}^{b} {\int\limits_{c}^{d} {f ({t,s})dsdt} }- \frac{1}{2} \Biggl[{ ({d - c}) \int\limits_{a}^{b} { \bigl[{f ({t,c}) + f ({t,d}) } \bigr] } dt} \\ &\quad \quad {} + ({b - a}) \int\limits_{c}^{d} { \bigl[{f ({a,s}) + f ({b,s}) } \bigr] ds} \Biggr] \\ &\quad \quad {}+ \frac{1}{4} ({b - a}) ({d - c}) \bigl[f ({a,c}) + f ({a,d}) + f ({b,c}) + f ({b,d}) \bigr] \\ &\quad \le\frac{1}{4} ({b - a}) ({d - c})\int\limits_{a}^{b} {\int\limits_{c}^{d} { \bigl|{D_{2} D_{1}f ({t,s}) } \bigr| } } dtds, \end{aligned}$$

which is Continuous Trapezoid type inequality

Corollary 2

(Discrete Case)

If\(\mathbb{T}_{1}=\mathbb{T}_{2}=\mathbb{Z}\)*then we get*

$$\begin{aligned} & \Biggl|{\sum\limits_{t = 1}^{k} {\sum\limits_{s = 1}^{r} {f ({t,s}) } }} - \frac{1}{2} \Biggl[m\sum\limits_{t = 1}^{k} { \bigl[{f ({t,1}) + f ({t,m + 1})} \bigr] } + k\sum\limits_{s = 1}^{r} \bigl[{f({1,s}) + f({k + 1,s})}\bigr]\Biggr] \Biggr| \\ &\quad {}\le \frac{1}{4}km \bigl[{f ({1,1}) + f ({1,r + 1}) + f ({k + 1,1}) + f ({k + 1,r + 1})} \bigr], \end{aligned}$$

which is Discrete Trapezoid type inequality.

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