Alpha Power Transformed Extended power Lindley Distribution

Research Article Open access Published: 09 December 2022

Volume 22, pages 1–18, (2023) Cite this article

Journal of Statistical Theory and Applications

Aims and scope

Submit manuscript

Fatehi Yahya Eissa & Chhaya Dhanraj Sonar

Abstract

The Lindley distribution has been generalized by several researchers in recent years. In this paper, we introduce and study a new generalization of extended power Lindley distribution named alpha power transformed extended power Lindley (*APTEPL*) distribution that provides better fits than the extended Power Lindley distribution and existing generalizations. It includes the alpha power transformed power Lindley, alpha power transformed extended

Lindley, alpha power transformed Lindley, extended power Lindley, power Lindley, extended Lindley and Lindley distribution as a special cases. In this article various properties of the *APTEPL* distribution such as moments, moment generating function, Characteristic function and cumulant generating function, quantiles and Order Statistics are derived. Method of maximum likelihood estimation is used to obtain the model parameters. A simulation study is performed to examine the performance of the maximum likelihood estimators of the parameters. Two data sets have been utilized to show how the *APTEPL* distribution works in practice.

1 Introduction

The power Lindley (*PL*) distribution and two-parameter Lindley (*TPL*) distribution were introduced by [8] and [17] respectively, these distributions are an extension to a known Lindley distribution which was proposed by [14]. After two years [2] introduced extended power Lindley distribution whose *CDF* and *PDF* are, respectively, given by

$$\begin{aligned} F(x)=1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }} \,\,\,\, ,\,\, x>0 ;\,\, \beta , \delta , \theta >0 \end{aligned}$$

(1)

$$\begin{aligned} f(x)=\frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }} \,\,\,\, ,\,\, x>0 ; \,\, \beta , \delta , \theta >0 \end{aligned}$$

(2)

Several methods of generating new statistical distributions were presented in the literature review such as, [16, 4], and [3] for more information can you see [12] and [11]. Another important method for generation a new distribution was proposed by [15] named alpha power transformed (*APT*). Its CDF and PDF are given as:

$$\begin{aligned} G(x,\alpha ,\xi)= {\left\{ \begin{array}{ll} \frac{\alpha ^{F(x,\xi)} -1}{\alpha -1}\,\, ,&{}\text{ if }\,\,\alpha >0,\,\,\alpha \ne 1 \\ F(x,\xi)\,\, , &{}\text{ if }\,\, \alpha = 1 \end{array}\right. } \end{aligned}$$

(3)

and

$$\begin{aligned} g(x,\alpha ,\xi)= {\left\{ \begin{array}{ll} \frac{\log \alpha }{\alpha -1} f(x) \alpha ^{F(x,\xi)}\,\, ,&{}\text{ if }\,\,\alpha \ne 1\\ f(x) \,\, , &{}\text{ if }\,\,\alpha = 1 \end{array}\right. } \end{aligned}$$

(4)

where $F(x)$ and $f(x)$ be the *CDF* and *PDF* of a random variable x and $x>0$.

Several authors utilized APT method to re-extend Lindley distributions in particular cases; for example, [5] introduced the alpha power transformed Lindley (*APTL*) distribution. Also, [6] presented alpha power transformed inverse Lindley (APTIL) distribution. Whereas alpha power transformed power Lindley (APTPL) distribution was introduced by [9]. Moreover, [7] proposed alpha power transformed power inverse Lindley (APTPIL) distribution.

This study aims to introduce a new lifetime distribution, referred to alpha power transformed extended power Lindley (APTEPL) distribution using the alpha power transformed method to the extended power Lindley (EPL) distribution. This model will be more flexibility in analysing the lifetime data. We are motivated to introduce the (*APTEPL*) distribution because: (1) It includes a number of well-known lifetime sub-models. (2) It can simulate monotonically increasing, decreasing, constant, bathtub, upside-down bathtub, and increasing - decreasing - increasing hazard rates. and (3) It can be viewed as a suitable model for fitting skewed data that may not be properly fitted by other common distributions, and it can also be used to solve a variety of problems in various fields, such as public health, biomedical studies, and industrial reliability and survival analysis.

The paper is organized as follows: Sect. 1 is introduction. In Sect. 2, the alpha power transformed extended power Lindley (*APTEPL*) distribution is defined. Some properties of *APTEPL* distribution are derived in Sect. 3 which includes: r^{th} moment, moment generating function, Characteristic function , cumulant generating function, ordered statistics and quantiles. The estimation of the unknown parameters by using maximum likelihood estimator are studied in Sect. 4. In Sect. 5, a simulation study is conducted to evaluate the performance of the different estimators. Finally, in Sect. 6 comparison the performance of proposed distribution with other distributions is verified using two real data sets, the first data set represent the waiting time (in minutes) of 100 bank customers and second data set consists of 128 bladder cancer patients.

2 Alpha Power Transformed Extended Power Lindley Distribution

Let $X\in R^+$ a random variable from extended power Lindley (*EPL*) distribution [2] with the scale parameter $\theta >0$ and shape parameters $\beta ,\delta >0$. By substituting the equations (2) and (1) into (3) and (4), then the *CDF* and *PDF* for *APTEPL* distribution given in the following expression:

$$\begin{aligned} G(x,\alpha ,\beta ,\delta ,\theta)= {\left\{ \begin{array}{ll} \frac{\alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}} -1}{\alpha -1}\,\,\, , &{} \text{ if } \alpha \ne 1 \\ \\ 1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\,\,\,, &{} \text{ if } \alpha =1 \end{array}\right. } \end{aligned}$$

(5)

and

$$\begin{aligned} g(x,\alpha ,\beta ,\delta ,\theta)= {\left\{ \begin{array}{ll} \frac{\log \alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }} \alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}, &{} \text{ if } \alpha \ne 1 \\ \\ \frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }}\,\,\,, &{} \text{ if } \alpha =1 \end{array}\right. } \end{aligned}$$

(6)

The corresponding survival function $S(x)$ and hazard rate function $h(x)$ respectively, as follow:

$$\begin{aligned} S(x,\alpha ,\beta ,\theta)= {\left\{ \begin{array}{ll} \frac{\alpha }{\alpha -1}\left(1-\alpha ^{-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}\right) \,\,, &{} \text{ if } \alpha \ne 1 \\ \\ \left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\,\,, &{} \text{ if } \alpha =1 \end{array}\right. } \end{aligned}$$

(7)

and

\$\$\begin{aligned} h(x,\alpha ,\beta ,\theta)= {\left\{ \begin{array}{ll} \frac{\theta ^2 \delta \log \alpha }{\theta +\beta }.\frac{x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }}}{\alpha ^{\left(1+\frac{\theta \beta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta

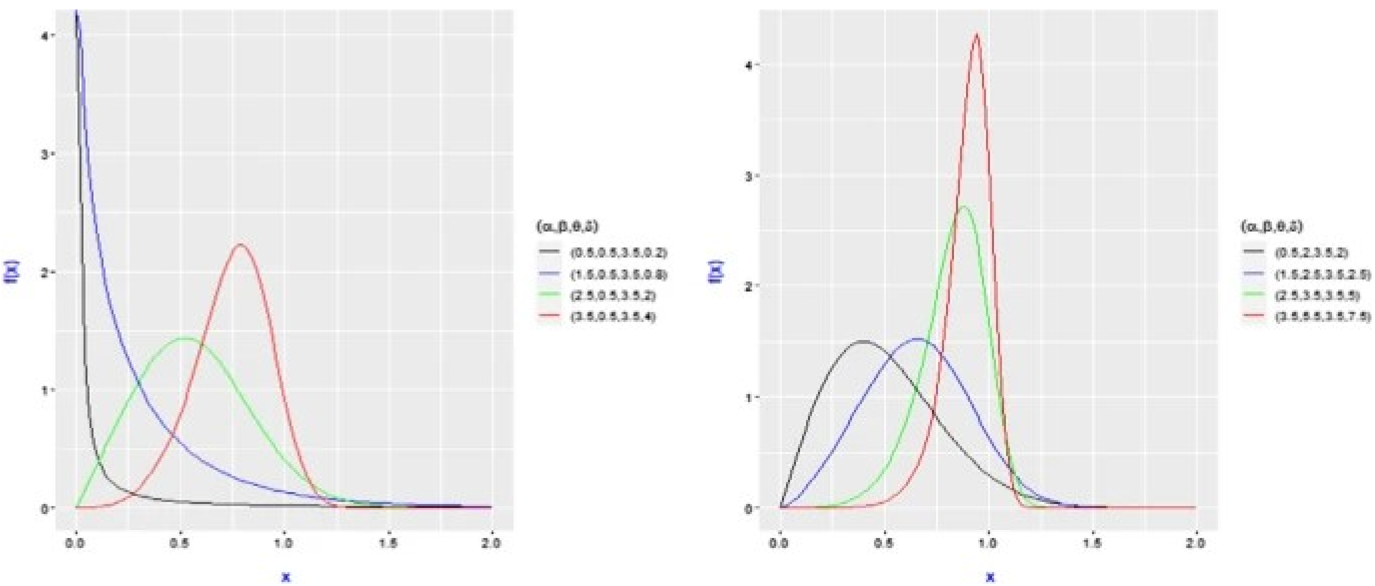
}}}-1}\,\,, &{} \text{ if } \alpha \ne 1 \\ \\ \frac{\theta ^2\delta x^{\delta -1}(1+\beta x^{\delta })}{\theta +\beta +\theta \beta x^{\delta }}\,\, , &{} \text{ if } \alpha =1 \end{array}\right. } \end{aligned}$$

(8)

A random variable *X* that follows alpha power transformed extended power Lindley distribution in (6) was denoted by \(X\sim APTEPL(x,\alpha ,\beta ,\delta ,\theta)\).

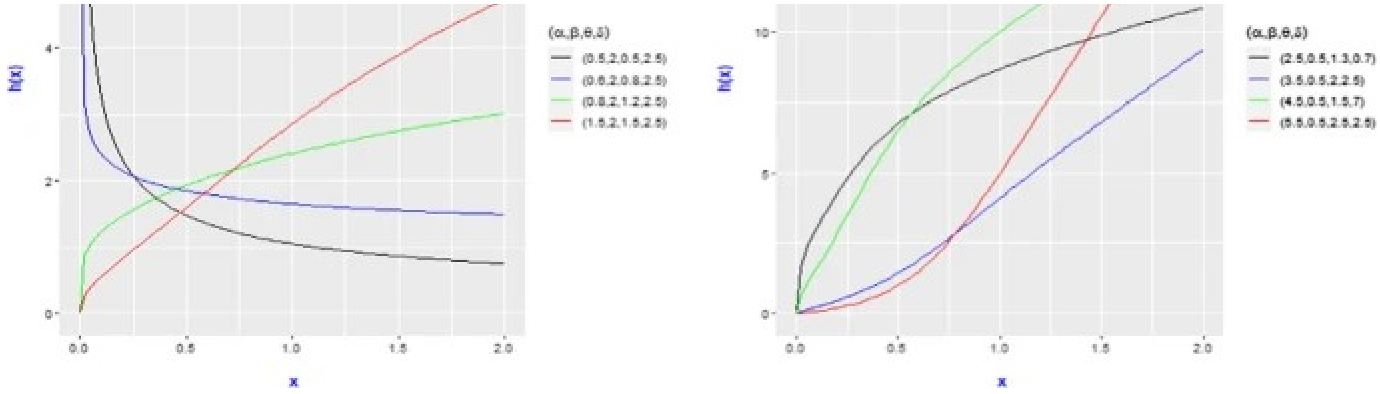
The *PDF* and *HRF* plots of \(X\sim APTEPL(x,\alpha ,\beta ,\delta ,\theta)\) are presented in Figs. 1 and 2 respectively,

Fig. 1



Plots of the *PDF* of the *APTEPLD* for different Values of Its Parameters

Fig. 2



Plots of the *hrf* of the *APTEPLD* for different Values of Its Parameters

---

Also, the reversed hazard rate function $r(x)$ and the cumulative hazard rate function $H(x)$ of the *APTEPL* distribution are, respectively, given as follows:

$$\begin{aligned} r(x)= {\left\{ \begin{array}{ll} \frac{\frac{\log \alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }} \alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}}{\frac{\alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}} -1}{\alpha -1}} , &{} \text{ if } \alpha \ne 1 \\ \\ \frac{\frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }}}{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}\,\,\,, &{} \text{ if } \alpha =1 \end{array}\right. } \end{aligned}$$

(9)

and

$$\begin{aligned} H(x)= {\left\{ \begin{array}{ll} -\log \left[\frac{\alpha }{\alpha -1}\left(1-\alpha ^{-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}\right) \right] \,\,, &{} \text{ if } \alpha \ne 1 \\ \\ - \log \left[\left(1+ \frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\right] \,\,, &{} \text{ if } \alpha =1 \end{array}\right. } \end{aligned}$$

(10)

2.1 Special Cases

Some well-known distributions are special cases of the *APTEPL* distribution. We present these cases for selected values of parameters.

- If \(\alpha =1\) the *APTEPL* distribution reduces to extended power Lindley (*EPL*) distribution [2].
- If \(\delta = 1\) the *APTEPL* distribution reduces to alpha power transformed extended Lindley (*APTEL*) distribution. (new)
- If \(\beta = 1\) the *APTEPL* distribution reduces to alpha power transformed power Lindley (*APTPL*) distribution [9].
- If \(\beta = \delta = 1\) the *APTEPL* distribution reduces to alpha power transformed Lindley (*APTL*) distribution [5].
- If \(\alpha = \beta = 1\) the *APTEPL* distribution reduces to power Lindley (*PL*) distribution [8].
- If \(\alpha = \delta = 1\) the *APTEPL* distribution reduces to two-parameter Lindley (*TPL*) distribution [17]
- If \(\alpha = \beta = \delta = 1\) the *APTEPL* distribution reduces to the well known Lindley (*L*) distribution [14].

Table 1 shows the specific values of the parameters used to generate the special cases of *APTEPL* distribution.

Table 1 The spacial cases of the APTEPL distribution

Note: Some spacial cases for the hazard rate function arising from the *APTEPL* distribution by assigning relevant values of the parameters.

3 Some Mathematical Properties of APTEPL Distribution

This section includes some properties of the *APTEPL* distribution like, r^{th} moment, moment generating function, characteristic function, cumulant generating function, and order statistics. and obtain the mean, standard deviation, skewness, kurtosis, and coefficients of variation.

3.1 r^{th} Moment

The r^{th} moment of *APTEPL* distribution is

$$\begin{aligned} {\mu _{r}^{'}}=&E(x^r)=\int _{0}^{\infty } x^{r} f(x,\alpha ,\beta , \theta) \,\, dx\nonumber \\ =&\frac{\log \alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta + \beta }\int _{0}^{\infty }x^{r} x^{\delta -1} (1+\beta x^{\delta })e^{-\theta x^{\delta }} \alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}\,\, dx \nonumber \\ =&\frac{\alpha \log \alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta +\beta }\int _{0}^{\infty } x^{r+\delta -1} (1+\beta x^{\delta })e^{-\theta x^{\delta }} \alpha ^{-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}\,\, dx \end{aligned}$$

(11)

Using power series ,

$$\begin{aligned} \alpha ^{-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}=\sum _{k=0}^{\infty }(-1)^{k}\frac{(\log \alpha)^{k}\left(1+ \frac{\beta \theta x^{\delta }}{\theta +\beta }\right) ^{k}e^{-\theta k x^{\delta }}}{k!} \end{aligned}$$

Therefore, (11) can be written as follows:

$$\begin{aligned} {\mu _{r}^{'}}&=\frac{\alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta + \beta }\sum _{k=0}^{\infty }\frac{(-1)^{k}(\log \alpha)^{k+1}}{k!} \nonumber \\ {} &\times \int _{0}^{\infty }x^{r+\delta -1}(1+\beta x^{\delta })e^{-\theta (1+k)x^{\delta

}}\left(1+\frac{\theta \beta x^{\delta }}{\theta +\beta }\right) ^{k}\,\,dx \end{aligned}$$

(12)

Also, by using series of Taylor,

$$\begin{aligned} \left(1+\frac{\theta \beta x^{\delta }}{\theta +\beta }\right) ^{k}=\sum _{l=0}^{k}\left({\begin{array}{c}k\\ l\end{array}}\right) \left(\frac{\theta \beta x^{\delta }}{\theta +\beta }\right) ^{l} \end{aligned}$$

Hence (12) can be expressed as follows:

$$\begin{aligned} {\mu _{r}^{'}}=\frac{\alpha \delta }{\alpha -1}\sum _{k=0}^{\infty }\sum _{l=0}^{k} C_{k,l}\int _{0}^{\infty }x^{r+\delta +\delta l-1}(1+\beta x^{\delta })e^{-\theta (1+k)x^{\delta }}\,\,dx \end{aligned}$$

After some steps we found that

$$\begin{aligned} {\mu _{r}^{'}}=\frac{\alpha }{\alpha -1}\sum _{k=0}^{\infty }\sum _{l=0}^{k} C_{k,l}\left[\frac{\Gamma (\frac{r}{\delta }+l+1)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+1}}+\beta \frac{\Gamma (\frac{r}{\delta }+l+2)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+2}}\right] \end{aligned}$$

(13)

where :

$$\begin{aligned} C_{k,l}=(-1)^{k} \beta ^{l} \left({\begin{array}{c}k\\ l\end{array}}\right) \frac{(\log \alpha)^{k+1}}{k!}\frac{\theta ^{l+2}}{(\theta +\beta)^{l+1}} \end{aligned}$$

Remark: the *mean* of *APTEPL* distribution is \(E(X)=\mu ={\mu _{1}^{'}}\) also, The variance, skewness, kurtosis and coefficient of variation can be obtained by: \(variance= \sigma ^2=\mu _{2}^{'}-\mu ^{2}\) ; \(\,\,\, skewness=\frac{\mu _{3}}{\sigma ^{3}}

$=\frac{\mu _{3}^{'}-3\mu _{2}^{'} \mu +2 \mu ^{3}}{(\mu _{2}^{'}- \mu ^{2})^{(3/2)}}$; $kurtosis=\frac{\mu _{4}}{\sigma ^{2}} =\frac{\mu _{4}^{'}-4\mu _{3}^{'} \mu +6\mu _{2}^{'}\mu ^{2}-2\mu ^{3}}{(\mu _{2}^{'}-\mu ^{2})^{2}}$ and $CV= \frac{\sigma }{\mu }=\sqrt{\frac{\mu _{2}^{'}}{\mu ^{2}}-1}$

3.2 Moment Generating Function, Characteristic Function and Cumulant Generating Function

- The moment generating function is given by

$$\begin{aligned} M_{X}(t)&=E(e^{tx})\nonumber \\ {}&=\int _{0}^{\infty }e^{tx} f(x)\,\,df\nonumber \\&=\sum _{r=0}^{\infty }\frac{t^{r}}{r!}\int _{0}^{\infty }x^{r} f(x,\alpha ,\beta ,\theta) \,\, dx\nonumber \\&=\sum _{r=0}^{\infty }\frac{t^{r}}{r!}\mu _{r}^{'} \end{aligned}$$

(14)

In case of using the equations (13) and (14) we found that moment generating function of *APTEPL* distribution can be expressed as follows:

$$\begin{aligned} M_{X}(t)=\frac{\alpha }{\alpha -1}\sum _{k=0}^{\infty }\sum _{l=0}^{k} C_{k,l}\frac{t^{r}}{r!}\left[\frac{\Gamma (\frac{r}{\delta }+l+1)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+1}}+\beta \frac{\Gamma (\frac{r}{\delta }+l+2)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+2}}\right] \end{aligned}$$

(15)

- The Characteristic function is given by

$$\begin{aligned} \phi _{X}(t)=E(e^{itx})=\int _{0}^{\infty }e^{itx}f(x)\,\,df \end{aligned}$$

similarly, after some steps the Characteristic function can be expressed:

$$\begin{aligned} \phi _{X}(t)=\frac{\alpha }{\alpha -1}\sum _{k=0}^{\infty }\sum _{l=0}^{k} C_{k,l}\frac{(it)^{r}}{r!}\left[\frac{\Gamma (\frac{r}{\delta }+l+1)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+1}}+\beta \frac{\Gamma (\frac{r}{\delta }

}+l+2)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+2}}\right] \end{aligned}$$

(16)

where $i=\sqrt{-1}$

- The Cumulant generating function is given by:

$$\begin{aligned} K(t)=\log \phi _{X}(t) \end{aligned}$$

(17)

In case of using the equations (16) and (17) we found that cumulant generating function of *APTEPL* distribution can be expressed:

$$\begin{aligned} K(t)=\log \left(\frac{\alpha }{\alpha -1}\right) +\log C \end{aligned}$$

(18)

where:

$$\begin{aligned} C=\sum _{k=0}^{\infty }\sum _{l=0}^{k} C_{k,l}\frac{(it)^{r}}{r!}\left[\frac{\Gamma (\frac{r}{\delta }+l+1)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+1}}+\beta \frac{\Gamma (\frac{r}{\delta }+l+2)}{\left[\theta (1+k)\right] ^{\frac{r}{\delta }+l+2}}\right] \end{aligned}$$

3.3 Quantile Function

The quantile function of the *APTEPL* distribution random variable *X* is $Q_{X}(u)=G_{X}^{-1}(u)$, $0<u<1$, and for any $\alpha ,\beta ,\delta ,\theta >0$ and $\alpha \ne 1$. In the following steps, we find out the expression of Q_{X}.

By considering the $p=1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}$, the CDF of *APTEPL* distribution (5) can be written as

$$\begin{aligned} G(x,\alpha , \beta , \delta , \theta)= \frac{\alpha ^p-1}{\alpha -1} \end{aligned}$$

(19)

By solving \(G_{X}(x)=u\) for *p*, we get

$$\begin{aligned} u&= \frac{\alpha ^p-1}{\alpha -1}\nonumber \\ u(\alpha -1)&=\alpha ^p-1\nonumber \\ \alpha ^p&=u(\alpha -1)+1\nonumber \\ p \log (\alpha)&=\log (u(\alpha -1)+1)\nonumber \\ p&=\frac{\log (u(\alpha -1)+1)}{\log (\alpha)} \end{aligned}$$

(20)

By solving \(p=1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\) for x, we obtain

$$\begin{aligned} p-1&=-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\nonumber \\ \left(p-1\right) \left(\frac{\theta +\beta }{\beta }\right)&=-\left(\frac{\theta +\beta }{\beta }+\theta x^{\delta }\right) e^{-\theta x^{\delta }}\nonumber \\ \left(p-1\right) \left(\frac{\theta +\beta }{\beta }\right) e^{-\left(\frac{\theta +\beta }{\beta }\right) }&=-\left(\frac{\theta +\beta }{\beta }+\theta x^{\delta }\right) e^{-\left(\frac{\theta +\beta }{\beta }+\theta x^{\delta }\right) } \end{aligned}$$

(21)

By using negative Lambert \(W_{-1}\) function, from (21) we get

$$\begin{aligned} -\left(\frac{\theta +\beta }{\beta }+\theta x^{\delta }\right) =W_{-1}\left(\left(p-1\right) \left(\frac{\theta +\beta }{\beta }\right) e^{-\left(\frac{\theta +\beta }{\beta }\right) }\right) \end{aligned}$$

(22)

From (22) and put \(Q_{X}(u)=x\), we obtain

$$\begin{aligned} Q_{X}(u)=\left\{ -\frac{1}{\beta }-\frac{1}{\theta }-\frac{1}{\theta } W_{-1}\left(\left(p-1\right) \left(\frac{\theta +\beta }{\beta }\right) e^{-\left(\frac{\theta +\beta }{\beta }\right) }\right) \right\} ^{\frac{1}{\delta }} \end{aligned}$$

(23)

By substituting (20) into (23), then the quantile function $Q_{X}(u)$ is given as

$$\begin{aligned} Q(u)=\left\{ -\frac{1}{\beta }-\frac{1}{\theta }-\frac{1}{\theta } W_{-1}\left(\left[\frac{\log (u(\alpha -1)+1)}{\log (\alpha)}-1\right] \left(\frac{\theta }{\beta }+1\right) e^{-\left(\frac{\theta }{\beta }+1\right) }\right) \right\} ^{\frac{1}{\delta }} \end{aligned}$$

(24)

whereas $W_{-1}(.)$ is the negative Lambert *W* function and $u\in (0,1)$. Further, the first quantile $Q1$ obtained by substituting $u=0.25$ in (24). The median $Q2$ and third quantile $Q3$ obtained in a similar manner. Tables 2 and 3 show behavior the first three quartiles of *APTEPL* distribution with the different values of Its parameters. We observed that if value of $\alpha ,\,\beta , \, \text {and} \, \delta$ are constant,while value of θ increases, value of each of Q1, Q2 and Q3 decrease. But in the case of values of $\beta ,\,\delta , \, \text {and} \,\theta$ are fixed, values of Q1, Q2 and Q3 increase when value of α increases. Also, in the case of value of β increases, whereas values of $\alpha ,\,\delta , \, \text {and} \, \theta$ are fixed , values of each of Q1, Q2 and Q3 increase too. Finally, values of each of Q1, Q2 and Q3 increase when value of δ increases, while value of $\alpha ,\, \beta , \,\text {and}\, \theta$ are constant.

Table 2 1^{st} quantile (Q1), median (Q2) and 3^{rd} quantile (Q3) when $\delta =0.5$ and different values of the other parameters of APTEPL distribution

Table 3 1^{st} quantile (Q1), median (Q2) and 3^{rd} quantile (Q3) when δ

=2\) and different values of the other parameters of APTEPL distribution

---

3.4 Order Statistics

Let $X_{(i)}, i =1,2,\ldots, n$ denoted to *n* independent random variables from any distribution with *CDF* $F_{X} (x)$ and *PDF* $f_{X} (x)$, then the *PDF* of $X_{(i)}$ is given by :

$$\begin{aligned} f_{X_{(i)}}(x)=\frac{n!}{(i-1)!(n-i)!} f_{X}(x)\left[F_{X}(x)\right] ^{i-1}\left[1-F_{X}(x)\right] ^{n-i}\,\,\,;i=1,2,..,n \end{aligned}$$

(25)

In case of substituting (6) and (5) into equation (25), the *PDF* of X_{i} according to *APTEPL* distribution given as the following:

$$\begin{aligned} f_{X_{(i)}}(x)=\frac{n!}{(i-1)!(n-i)!} \frac{\log \alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }} \alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}}\nonumber \\ \,\,\,\times \left[\frac{\alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}} -1}{\alpha -1}\right] ^{i-1}\times \left[1-\frac{\alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}} -1}{\alpha -1}\right] ^{n-i} \end{aligned}$$

(26)

Now, the *PDF* of $X_{(n)}$ and $X_{(1)}$ respectively, given by :

$$\begin{aligned} f_{X_{(n)}}(x)=n\,\frac{\log \alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }} \alpha ^{\left[1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\right] }\nonumber \\ \,\,\,\times \left[\frac{\alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}} -1}{\alpha -1}\right] ^{n-1} \end{aligned}$$

(27)

and

$$\begin{aligned} f_{X_{(1)}}(x)=n\,\frac{\log \alpha }{\alpha -1}\frac{\theta ^2\delta }{\theta +\beta }x^{\delta -1}(1+\beta x^{\delta })e^{-\theta x^{\delta }} \alpha ^{\left[1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\right] }\nonumber \\\,\,\,\times \left[1-\frac{\alpha ^{1-\left(1+\frac{\beta \theta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}} -1}{\alpha -1}\right] ^{n-1} \end{aligned}$$

(28)

4 Simulation Study

The simulation study for *MLEs* of *APTEPL* distribution is performed by generating $N=1000$ samples of sizes $n=20, 40,\,\,80,\,\, and \,\, 100$ from *APTEPL* distribution and studied the behavior of estimates, based on certain measures, which are mean square errors (*MSEs*) and absolute biases (*ABs*). Considering *CDF* of *APTEPL* distribution and after performing mathematical calculations it was found that if u is a random number from $U(0, 1)$, then

$$\begin{aligned} x=\left\{ -\frac{1}{\beta }-\frac{1}{\theta }-\frac{1}{\theta }W_{-1}\left(\left[\frac{\log (u(\alpha -1)+1)}{\log \alpha }-1\right] \left(\frac{\theta +\beta }{\beta }\right) e^{-\left(\frac{\theta }{\beta }+1\right) }\right) \right\} ^\frac{1}{\delta } \end{aligned}$$

(29)

of the *APTEPL* distribution with parameters $\alpha ,\beta ,\delta ,$ and θ. where $W_{-1}(.)$ is the negative Lambert *W* function.

Tables 4 and 5 show the empirical results of a simulation study using *MLEs* and the *goodness.fit*() function in the \('AdequacyModel'\) package via the R program. The *MSEs* and *ABs* decrease as the sample size increases.

Table 4 Simulation results: ***MSEs*** **and** ***ABs***

Table 5 Simulation results: ***MSEs*** **and** ***ABs***

5 Maximum Likelihood Estimator

Let \(X= (x_{1}, x_{2}, ..., x_{n})\) be the random variables of size *n* belong to \(APTEPL(x,\alpha ,\beta ,\delta , \theta)\), then the Likelihood function can be written as,

$$\begin{aligned} L=\prod _{i=1}^{n}f(x, \alpha ,\beta , \delta ,\theta) \end{aligned}$$

(30)

Substituting (6) into (30) and taking the logarithm function for two side we found that the \(Loglikelihood\,\, function\) is given by:

$$\begin{aligned} \log L&= n\log (\log (\alpha))+2n\log \theta +n\log \delta -n\log (\alpha -1)-n\log (\theta +\beta)\nonumber \\ {}&+\sum _{i=1}^{n}\log x^{\delta -1}+\sum _{i=1}^{n}\log (1+\beta x^{\delta })-\theta \sum _{i=1}^{n}x^{\delta }\nonumber \\ {}&+\sum _{i=1}^{n}\left[1-\left(1+\frac{\theta \beta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\right] \log (\alpha) \end{aligned}$$

(31)

By taking the partial derivative for Log L in (31) with respected to $\alpha ,\theta , \beta \, \text {and} \, \delta$ respectively , we got that

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha }&=\frac{n}{\alpha \log \alpha }-\frac{n}{\alpha -1}+\frac{1}{\alpha }\sum _{i=1}^{n}\left[1-\left(1+\frac{\theta \beta x^{\delta }}{\theta +\beta }\right) e^{-\theta x^{\delta }}\right] \end{aligned}$$

(32)

$$\begin{aligned} \frac{\partial \log L}{\partial \theta }&=\frac{2n}{\theta }- \frac{n}{\theta +\beta }-\sum _{i=1}^{n} x^{\delta } +\log (\alpha) \sum _{i=1}^{n}\left[1-\left(\frac{\beta }{\theta +\beta }\right) ^{2}+\frac{\theta \beta x^{\delta }}{\theta +\beta }\right] x^{\delta }e^{-\theta x^{\delta }} \end{aligned}$$

(33)

$$\begin{aligned} \frac{\partial \log L}{\partial \beta }&=-\frac{n}{\theta +\beta }+\sum _{i=1}^{n}\frac{x^{\delta }}{1+\beta x^{\delta }}+\log (\alpha)\,\,\sum _{i=1}^{n}\left[\frac{\theta ^2 x^{\delta }}{(\theta +\beta)^2}e^{-\theta x^{\delta }}\right] \end{aligned}$$

(34)

$$\begin{aligned} \frac{\partial \log L}{\partial \delta }&=\frac{n}{\delta }+\sum _{i=1}^{n}\frac{x^{\delta -1}\log x}{x^{\delta -1}}+\sum _{i=1}^{n}\frac{\beta x^{\delta }\log x}{1+\beta x^{\delta }}-\theta \sum _{i=1}^{n}x^{\delta }\log x \nonumber \\&+ \log (\alpha)\,\,\sum _{i=1}^{n}\left[\frac{1+\theta \beta x^{\delta }}{\theta +\beta }\theta ^2 x^{\delta } e^{-\theta x^{\delta }} \log x \right] \end{aligned}$$

(35)

The maximum likelihood estimator of α ,θ , β and δ can be obtained by solving the above four nonlinear equations when $\frac{\partial \log L}{\partial \alpha }=0$, $\frac{\partial \log L}{\partial \theta }=0$, \(\frac{\partial \log L}{\partial

\beta }=0\) and $\frac{\partial \log L}{\partial \delta }=0$ with using the numerical approach.

6 Application

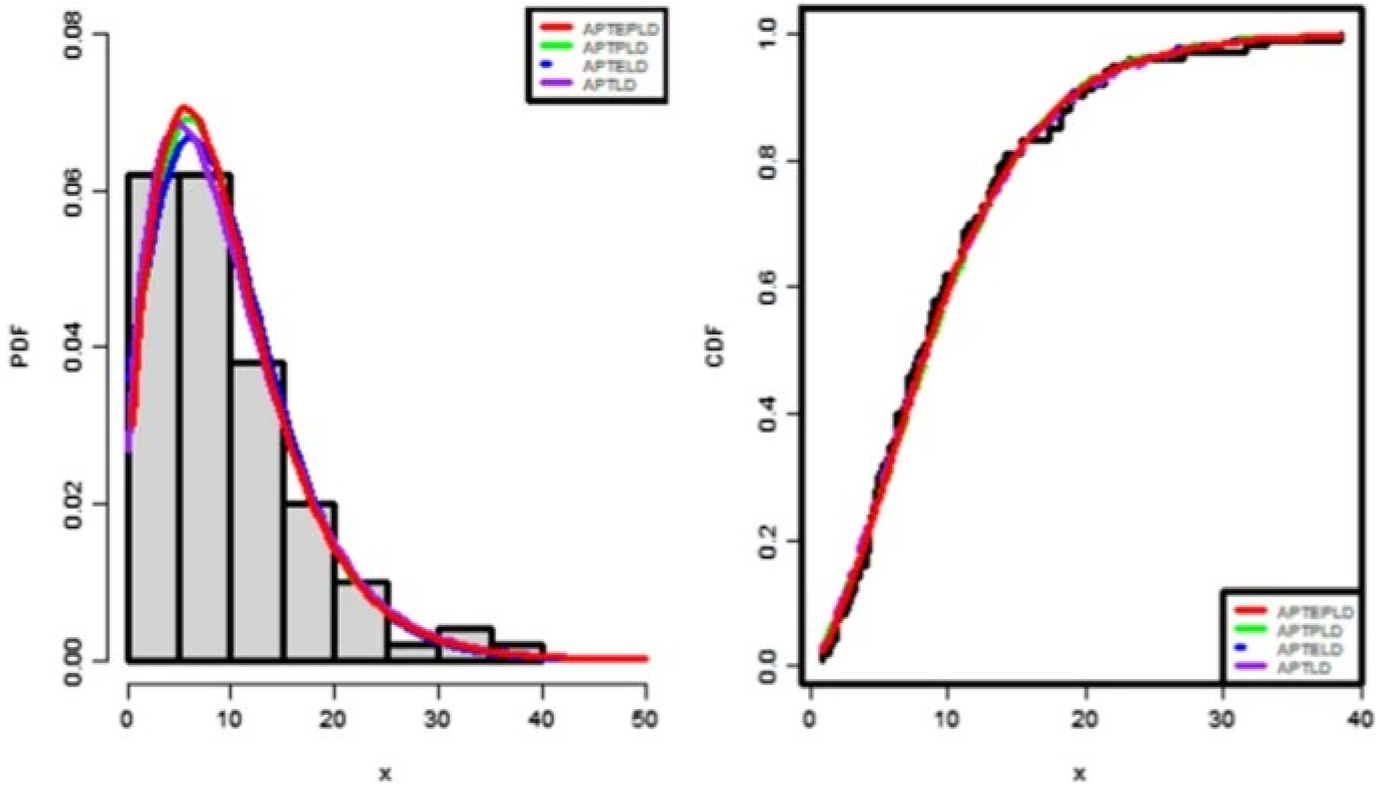
In this section, two real data sets are used to compare the performance of proposed APTEPL distribution with three existing models: alpha power transformed power Lindley (*APTPL*) distribution [9] , alpha power transformed extend Lindley (*APTEL*) which is as spacial case of our proposed distribution and alpha power transformed Lindley (*APTL*) distribution [5]. To compare the performance of our model with the others models the following criterions are used: Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*), Corrected Akaike information criterion (*AICc*), and Kolmogorov-Smirnov goodness of fit test (*KS*). The distribution with the smallest values of $AIC, BIC,\,\, \text {and} \,\, AICc$ or maximum $p-value$ for (*KS*) test is considering as the best model for the given data. In this section the numerical results are obtained by using of R software.

First data set taken from Ijaz Muhammad et al. [10]. The data set represent the waiting time (in minutes) of 100 bank customers. The MLE of the parameters and standard errors and goodness of fit statistics of the model parameters are provided in Tables 6 and 7. It is clear from Table 7 that the APTEPL distribution gives better fit than the other distributions. Figure 3 shows the theoretical and empirical $PDF\,\, \text {and} \,\,CDF$ of the $APTEPL, APTPL, APTEL,\,\, \text {and} \,\, APTL$ distributions using Bank customer data.

Table 6 ***MLEs*** **and** ***SE*** **of parameters for data set 1**

Table 7 Measure of $-logL,\, AIC,\, BIC,\, AICc,\, KS \,\, \text {and} \,\, P-value (KS)$ for data set 1

Fig. 3



plots of the estimated PDFs and CDFs of APTEPL distribution and other distributions for data set 1

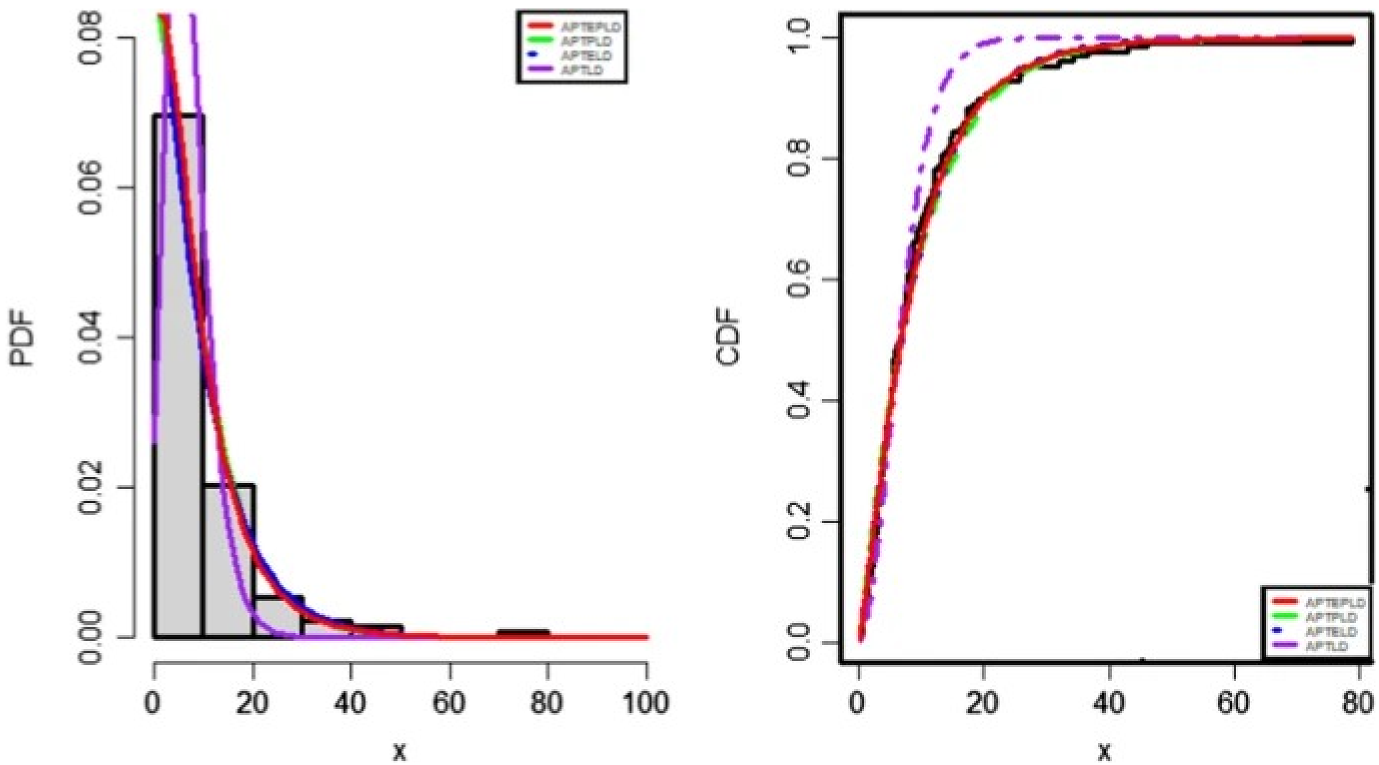
Second data set consists of 128 bladder cancer patients taken from Lee Elisa et al. [13] and used by Ahmad et al. [1]. The *MLEs* of the parameters and standard errors and goodness of fit statistics of the model parameters are provided in Table 8 and 9, from Table 9 it can be that the *APTEPL* distribution gives better fit than the other distributions. Figure 4 shows the theoretical and empirical *PDF* and *CDF* of the \(APTEPL, APTPL, APTEL,\,\, \text {and} \,\, APTL\) distributions regarding data set 2.

Table 8 *MLEs* and *SE* of parameters for data set 2

Table 9 Measure of \(-logL,\, AIC,\, BIC,\, AICc,\, KS \,\, \text {and}\,\, P-value(KS)\) for set

2

Fig. 4



plots of the estimated PDFs and CDFs of APTEPL distribution and other distributions for data set 2

7 Conclusion

In this article, we have proposed alpha power transformation extended power Lindley distribution which is extension of extended power Lindley distribution using alpha power transformation. Some properties of the proposed distribution are derived such as moments, moment generating function, Characteristic function , cumulant generating function, quantiles and order statistics. From the proposed distribution various distributions related to Lindley distribution can be obtained as a special cases of it. If \(\beta =1\) alpha power transformed power Lindley is obtained, similarly, if \(\delta =1\) alpha power transformed extended Lindley, if \(\beta =1\) and \(\delta =1\) alpha power transformed Lindley, if \(\alpha

$=1$ extended power Lindley, if $\alpha =1$ and $\beta =1$ power Lindley, if $\alpha =1$ and $\delta =1$ extended Lindley and if $\alpha =1$, $\beta =1$ and $\delta =1$ Lindley distribution are obtained. The performance of the proposed distribution is verified by using real data sets and performing simulation study. It was observed that the proposed distribution is good w.r.t the comparing criterions $AIC, BIC \,\, \text {and}\,\, AICc$. Therefore it was better than other models considered in this article. It works well for the real life data to model the waiting time (in minutes) of 100 bank customers data and the data of 128 bladder cancer patients.

Availability of data and material

The real data sets used in Applications: Data set 1 taken from Ijaz et al. (2021). Data set 2 taken from Lee and Wang (2003) and used by Ahmad et al. (2019).

Abbreviations

CDF : The cumulative distribution function

PDF : The probability density function

L : Lindley distribution

PL : Power Lindley distribution

EPL : Extended power Lindley distribution

TPL : Two parameter Lindley distribution

APT : Alpha power transformed

APTL : Alpha power transformed Lindley distribution

APTEL : Alpha power transformed extended Lindley distribution

APTPL : Alpha power transformed power Lindley distribution

APTEPL : Alpha power transformed extended power Lindley distribution

MLEs : Maximum likelihood estimators

AIC : Akaike information criterion

BIC : Bayesian information criterion

AICc : Corrected Akaike information criterion

KS : Kolmogorov-Smirnov goodness of fit test

$f(x)$: Probability density function

$F(x)$: Cumulative distribution function

$S(x)$: Survival function

$h(x)$: Hazard function

$H(x)$: Cumulative hazard function

$r(x)$: Reverse hazard rate function

\(E(x^r)\) : The *r*th moment of distribution.

\(M_{x}(t)\) : Moment generating function

\(\phi _{x}(t)\) : Characteristic function

K(t): Cumulant generating function

Q(u): Quantile function

\(W_{-1}(.)\) : Negative Lambert function

References

1. Ahmad, Z., Ilyas, M., Hamedani, G.G.: The extended alpha power transformed family of distributions: properties and applications. J. Data Sci. 17(4), 726–741 (2019)

   Article Google Scholar

2. Alkarni, S.H.: Extended power Lindley distribution: a new statistical model Fornon-monotone survival data. Eur. J. Stat. Probab. 3(3), 19–34 (2015)

   Google Scholar

3. Alzaatreh, A., Lee, C., Famoye, F.: A new method for generating families of continuous distributions. Metron 71(1), 63–79 (2013)

   Article MathSciNet MATH Google Scholar

4. Cordeiro, G.M., de Castro, M.: A new family of generalized distributions. J. Stat. Comput. Simul. 81(7), 883–898 (2011)

   Article MathSciNet MATH Google Scholar

5. Dey, S., Ghosh, I., Kumar, D.: Alpha-power transformed Lindley distribution: properties and associated inference with application to earthquake data. Ann. Data Sci. 6(4), 623–650

(2019)

Article Google Scholar

6. Dey, S., Nassar, M., Kumar, D.: Alpha power transformed inverse Lindley distribution: a distribution with an upside-down bathtub-shaped hazard function. J. Comput. Appl. Math. 348, 130–145 (2019)

Article MathSciNet MATH Google Scholar

7. Eltehiwy, M.: On the alpha power transformed power inverse Lindley distribution. J. Indian Soc. Probab. Stat. 1–24 (2020)

8. Ghitany, M.E., Balakrishnan, N., Al-Enezi, L.J., Al-Mutairi, D.K.: Power Lindley distribution and associated inference. Comput. Stat. Data Anal. 64, 20–33 (2013)

Article MathSciNet MATH Google Scholar

9. Hassan, A.S., Elgarhy, M., Mohamd, R.E., Alrajhi, S.: On the alpha power transformed power Lindley distribution. J. Probab. Stat. 2019 (2019)

10. Ijaz, M., Mashwani, W.K., Göktaş, A., Unvan, Y.A.: A novel alpha power transformed exponential distribution with real-life applications. J. Appl. Stat. 1–16 (2021)

11. Jones, M.C.: On families of distributions with shape parameters. Int. Stat. Rev. 83(2), 175–192 (2015)

Article MathSciNet Google Scholar

12. Lee, C., Famoye, F., Alzaatreh, A.Y.: Methods for generating families of univariate continuous distributions in the recent decades. Wiley Interdiscip. Rev. Comput. Stati.

5(3), 219–238 (2013)

13. Lee, E.T., Wang, J.: Statistical Methods for Survival Data Analysis, vol. 476. Wiley, New York (2003)

Book MATH Google Scholar

14. Lindley, D.V.: Fiducial distributions and Bayes' theorem. J. R. Stat. Soc. Ser. B (Methodol.) 20(1), 102–107 (1958)

15. Mahdavi, A., Kundu, D.: A new method for generating distributions with an application to exponential distribution. Commun. Stat. Theory Methods 46(13), 6543–6557 (2017)

Article MathSciNet MATH Google Scholar

16. Marshall, A.W., Olkin, I.: A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. Biometrika 84(3), 641–652 (1997)

Article MathSciNet MATH Google Scholar

17. Shanker, R., Sharma, S., Shanker, R.: A two-parameter Lindley distribution for modeling waiting and survival times data (2013)

Acknowledgements

The authors would like to express their heartfelt appreciation to the Editors of the Journal of Statistical Theory and Applications for waiving the APC payment. The authors would also like to thank the editor and anonymous referees for their insightful comments and suggestions, which helped to improve the paper's quality.

Funding

No funding

Author information

Authors and Affiliations

Department of Mathematics, Faculty of Education and Applied Science, Amran University, Amran, Yemen
Fatehi Yahya Eissa

Department of Statistics, Faculty of Science and Technology, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, Maharashtra, India
Fatehi Yahya Eissa & Chhaya Dhanraj Sonar

Contributions

Conceptualization, F. E.; Writing - original draft preparation, F. E. ; review and editing, C. D. S.

Corresponding author

Correspondence to Fatehi Yahya Eissa.

Ethics declarations

Ethics approval and consent to participate:

This article is unique, contains previously unpublished material, and contains no ethics issues.

Conflicts of Interest

The authors declare no conflict of interest.

Rights and permissions

About this article

Cite this article

Eissa, F.Y., Sonar, C.D. Alpha Power Transformed Extended power Lindley Distribution. *J Stat Theory Appl* 22, 1–18 (2023). https://doi.org/10.1007/s44199-022-00051-3

Received	Accepted	Published
21 February 2022	23 November 2022	09 December 2022

Issue Date

June 2023

DOI

https://doi.org/10.1007/s44199-022-00051-3

Share this article

Anyone you share the following link with will be able to read this content:

Get shareable link

Provided by the Springer Nature SharedIt content-sharing initiative

Keywords

Alpha power transformed
Lindley distribution
Extended power Lindley distribution
Maximum likelihood estimators