Boundary Value Problems

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Some properties of implicit impulsive coupled system via φ-Hilfer fractional operator

Mohammed A. Almalahi & Satish K. Panchal

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Abstract

The major goal of this work is investigating sufficient conditions for the existence and uniqueness of solutions for implicit impulsive coupled system of φ-Hilfer fractional differential equations (FDEs) with instantaneous impulses and terminal conditions. First, we derive equivalent fractional integral equations of the proposed system. Next, by employing some standard fixed point theorems such as Leray–Schauder alternative uniqueness of solutions. Further, by mathematical analysis technique we investigate the Ulam–Hyers (UH) and generalized UH (GUH) stability of solutions. Finally, we provide a pertinent example to corroborate the results obtained.

1 Introduction

Fractional differential equations (FDEs) have attracted the interest of researchers from various disciplines as they are a useful tool in modeling the dynamics of numerous physical systems and have applications in many fields of applied sciences, engineering and technical sciences, and so on. For further details, see [26, 36, 38, 40]. There are various definitions of fractional calculus (FC) used in FDEs for modeling and describing the memory accurately. Among the famous operators of this calculus, there are Riemann–Liouville, Riemann, Grünwald–Letnikov, Caputo, Hilfer, and Hadamard, which are the most used. For more detail, we refer the readers to [1–3, 21, 22, 24, 25, 33, 34, 36, 41]. There is a prominent and noticeable interest in the investigation of qualitative characteristics of solutions (existence, uniqueness, stability) of FDEs. For applications and recent work, we refer the readers to [4, 7, 14, 18, 37, 42, 43].

In recent years, the impulsive fractional differential equations have become an important and successful tool in modeling some physical phenomena that have sudden changes and have discontinuous jumps by imposing impulsive conditions on the fractional differential equations at discontinuity points. For applications and recent work, we refer the readers to [8, 9, 12, 13, 17, 27, 28, 32, 44].

On the other side, the study of coupled systems involving FDEs is also important as such systems occur in various problems of applied nature. For some theoretical works on coupled systems of FDEs, we refer to series of papers [11, 16, 19, 20, 23, 30].

The topic of system stability is one of the most important qualitative characteristics of a solution, but to our knowledge, the results on UH and UHR stability of solutions for implicit impulsive coupled systems are very few in the literature.

Very recently, Kharade and Kucche [35] studied the existence and uniqueness of solutions and UHML stability for the following impulsive implicit problem:

$$\textstyle\begin{cases} \mathcal{D}_{a^{+}}^{\mathfrak{y},\mathfrak{p};\mathfrak{\varphi }}u(\sigma)=f(\sigma ,u(\sigma),u(h(\sigma)),\mathcal{D}_{a^{+}}^{ \mathfrak{y},\mathfrak{p};\mathfrak{\varphi }}y(\sigma)),&\sigma \in \mathcal{J}:= [0,T] ,\sigma \neq \sigma _{k},k=1,\ldots,p, \\ \Delta I_{0^{+}}^{1-\mathfrak{\gamma },\mathfrak{\varphi }}u(\sigma _{k})\vert _{\sigma =\sigma _{k}}=J_{k}u(\sigma _{k}^{-}),& k=1,\ldots,p,\\ I_{0^{+}}^{1- \mathfrak{\gamma },\mathfrak{\varphi }}u(0)=u_{0}, \\ u(\sigma)=\phi (\sigma),&\sigma \in [-r,0] , \end{cases}$$

where $\mathcal{D}_{a^{+}}^{\mathfrak{y},\mathfrak{p};\mathfrak{\varphi }}$ denotes the φ-Hilfer fractional derivative (FD) of order $\mathfrak{y}\in$

$f:\mathcal{J}\times \mathbb{R} \times$

\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} is a continuous function. Via standard fixed point theorems, Ahmed et al. [10] studied the existence, uniqueness, and different kinds of stability of the following switched coupled implicit φ-Hilfer fractional differential system:

\textstyle\begin{cases} \mathcal{D}_{a^{+}}^{\mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma)=f(\sigma ,\mathfrak{u}(\sigma),\mathcal{D}_{a^{+}}^{\mathfrak{y,p,\varphi }} \vartheta (\sigma)),&\sigma \in J:= [0,T], \\ \mathcal{D}_{a^{+}}^{\mathfrak{y,p,\varphi }}\vartheta (\sigma)=g(\sigma ,\mathcal{D}_{a^{+}}^{\mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma), \vartheta (\sigma)),&\sigma \in J:= [0,T] , \\ I_{a^{+}}^{1-\gamma ,\mathfrak{\varphi }}\mathfrak{u}(a)= \mathfrak{u}_{a},\qquad I_{a^{+}}^{1-\gamma ,\mathfrak{\varphi }}\vartheta (a)= \vartheta _{a},\mathfrak{u}_{a},&\vartheta _{a}\in \mathbb{R} ,\end{cases}

where \mathcal{D}_{a^{+}}^{\mathfrak{y,p,\varphi }} denotes the φ-Hilfer FD of order \mathfrak{y}\in (0,1) and type \mathfrak{p}\in [0,1] , and f,g: [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} are continuous functions.

Abdo et al. [5], via standard fixed point theorems, studied the existence and uniqueness of the following impulsive problem:

\textstyle\begin{cases} ^{AB}\mathcal{D}_{ [

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,u(\sigma)),&\sigma \in \mathcal{J}:= [0,T] ,

\sigma \neq \sigma _{k},k=1,\ldots,m, \\ \Delta u
\vert _{\sigma =\sigma _{k}}=I_{k}u(\sigma
_{k}^{-}),&k=1,\ldots,m,\\
u(0)=u_{0}.\end{cases}

On the other hand, Almalahi et al. [15] studied the existence and uniqueness of solution for the following FDEs:

\textstyle\begin{cases}
\mathcal{D}_{a^{+}}^{\mathfrak{y},\mathfrak{p};\mathfrak{\varphi
}}y(\sigma)=f (\sigma ,y(\sigma
),\mathcal{D}_{a^{+}}^{
\mathfrak{y},\mathfrak{p};\mathfrak{\varphi
}}y(\sigma)) ,&\sigma \in (a,T] ,a>0, \\
y(T)=w\in \mathbb{R} ,\end{cases}

where
\mathcal{D}_{a^{+}}^{\mathfrak{y},\mathfrak{p
};\mathfrak{\varphi }} is the φ-Hilfer FD of order
\mathfrak{y}\in (0,1) and type \mathfrak{p}\in [
0,1] .
Abdo et al. [6] studied the existence, uniqueness, and UH stability of the following system:

\textstyle\begin{cases}
\mathcal{D}_{a^{+}}^{\mathfrak{y}_{1},\mathfrak{p}_{1};
\mathfrak{\varphi }}y(\sigma)=f_{1} (\sigma
,y(\sigma)) ,& \sigma \in (a,T] ,a>0 \\
\mathcal{D}_{a^{+}}^{\mathfrak{y}_{2},\mathfrak{p}_{2};
\mathfrak{\varphi }}x(\sigma)=f_{2} (\sigma
,y(\sigma)) \\ y(T)=w_{1}\in \mathbb{R} , \\
x(T)=w_{2}\in \mathbb{R} , \end{cases}

where
\mathcal{D}_{a^{+}}^{\mathfrak{y}_{1},\mathfr
ak{p}; \mathfrak{\varphi }},
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ak{p}; \mathfrak{\varphi }} are the φ-Hilfer FDs of orders $\mathfrak{y}_{1},\mathfrak{y}_{2}\in (0,1)$ and type $\mathfrak{p}\in [0,1]$.

Motivated by the preceding works, in this paper, we investigate the existence, uniqueness, and UH stability for more general implicit impulsive coupled systems of φ-Hilfer FDEs:

$$\textstyle\begin{cases} \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \mathfrak{u}(\sigma)=f(\sigma ,\mathfrak{u}(\sigma),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta (\sigma)),\quad \sigma \in J:= [0,T] ,\sigma \neq \sigma _{k},k=1,\ldots,m, \\ \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \vartheta (\sigma)=g(\sigma ,\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma),\vartheta (\sigma)),\quad \sigma \in J:= [0,T] ,\sigma \neq \sigma _{k},k=1,\ldots,m, \\ \Delta \mathfrak{u} \vert _{\sigma =\sigma _{k}}=Z_{k} \mathfrak{u}(\sigma _{k}^{-}),\quad k=1,\ldots,m, \Delta \vartheta \vert _{\sigma =\sigma _{k}}=Z_{k}\vartheta (\sigma _{k}^{-}),k=1,\ldots,m, \\ \mathfrak{u} (T)=w_{1},\qquad \vartheta (T)=w_{2}, \end{cases}$$

(1.1)

where $\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}$ denotes the φ-Hilfer FD of order $\mathfrak{y}\in (0,1)$ and type $\mathfrak{p}\in [0,1]$, $[\sigma] =\sigma _{k}$ for $\sigma \in (\sigma _{k},\sigma _{k+1}]$, $k=0, 1,\ldots,m$, $\sigma _{0}=0$. The functions

$f:J\times \mathbb{R}\times \mathbb{R} \longrightarrow \mathbb{R}$ and

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$Z_{k}:\mathbb{R} \longrightarrow \mathbb{R}$, $k=1,2,\ldots,m$, are continuous functions fulfilling some conditions that will be described later. Further, $w_{1},w_{2}\in \mathcal{\mathbb{R} }$, $\sigma _{k}$ satisfy $0=\sigma _{0}<\sigma _{1} <\cdots <\sigma _{k}<\sigma _{k+1}=\sigma$, . $\Delta \mathfrak{u} \vert _{\sigma =\sigma _{k}}= \mathfrak{u}(\sigma _{k}^{+})- \mathfrak{u}(\sigma _{k}^{-})=\mathfrak{u} (\sigma _{k}^{+})-\mathfrak{u}(\sigma _{k})$, $\mathfrak{u}(\sigma _{k}^{+})=\lim_{h\rightarrow 0^{+}}\mathfrak{u}(\sigma _{k}+h)$, $\mathfrak{u}(\sigma _{k}^{-})=\lim_{h\rightarrow 0^{-}}\mathfrak{u} (\sigma _{k}+h)$ represent the right and left limits of $\mathfrak{u}(\sigma)$ at $\sigma \in (\sigma _{k},\sigma _{k+1}]$, $k=0,1,\ldots,m$, . $\Delta \vartheta \vert _{\sigma =\sigma _{k}}= \vartheta (\sigma _{k}^{+})-\vartheta (\sigma _{k}^{-})=\vartheta (\sigma _{k}^{+})-\vartheta (\sigma _{k})$, $\vartheta (\sigma _{k}^{+})=\lim_{h\rightarrow 0^{+}}\vartheta (\sigma _{k}+h)$ and $\vartheta (\sigma _{k}^{-})=\lim_{h\rightarrow 0^{-}}\vartheta (\sigma _{k}+h)$ represent the right and left limits of $\vartheta (\sigma)$ at $\sigma \in (\sigma _{k},\sigma _{k+1}]$, $k=0,1,\ldots,m$.

The coupled systems of φ-Hilfer FDEs with impulsive conditions considered in this work are a wider class of coupled systems of BVPs that incorporates the BVPs for FDEs involving the most broadly used Riemann–Liouville and Caputo fractional derivatives. Regardless of this, the

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function φ and parameter $\mathfrak{p}$ include coupled systems of FDEs involving the Hilfer, Hadamard, Katugampola, and many other fractional derivative operators.

- If $\varphi(\sigma)=\sigma$ and $\mathfrak{p}=1$, then system (1.1) reduces to an implicit impulsive coupled system with the Caputo fractional derivative.
- If $\varphi(\sigma)=\sigma$ and $\mathfrak{p}=0$, then system (1.1) reduces to an implicit impulsive coupled system with the Riemann–Liouville fractional derivative.
- If $\mathfrak{p}=0$, then system (1.1) reduces to an implicit impulsive coupled system with the φ-Riemann–Liouville fractional derivative.
- If $\varphi(\sigma)=\sigma$, then system (1.1) reduces to an implicit impulsive coupled system with the Hilfer fractional derivative.
- If $\varphi(\sigma)=\log \sigma$, then system (1.1) reduces to an implicit impulsive coupled system with the Hilfer–Hadamard fractional derivative.
- If $\varphi(\sigma)=\sigma^{\rho}$, then system (1.1) reduces to an implicit impulsive coupled system with the Katugampola fractional derivative.

The major contribution of this paper is obtaining an equivalent fractional integral equation of the proposed system and establishing the existence, uniqueness, and UH and GUH stability of a solution for an implicit impulsive coupled system with φ-Hilfer FD. Our analysis relies on the Banach and Leray–Schauder fixed point theorems. Though we use the standard methodology to obtain our results, its exposition to the proposed system is new. The

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general and cover many parallel problems that contain particular cases of functions because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives. Moreover, the results obtained in this work can be extended to *n*-tuple fractional systems (FSs). Our results include the results of Almalahi et al. [15], Abdo et al. [6], and Kharade et al. [35] and will be a useful contribution to the existing literature on this topic.

This paper is organized as follows. In Sect. 2, we render the rudimentary definitions and prove some lemmas and present some concepts of fixed point theorems. In Sect. 3, we prove the existence and uniqueness of solutions for impulsive implicit coupled system (1.1). In Sect. 4, we discuss the stability by means of mathematical analysis techniques. In Sect. 5, we give a pertinent example illustrating our results. Concluding remarks are presented in the last section.

2 Background material and auxiliary results

In this part, we give important definitions and auxiliary lemmas pertinent to our main results.

Let $J:= [0,T]$ and $J^{\prime }:= (0,T]$. Let $\mathcal{\mathbb{R} }=\mathcal{C} (J)$ be the Banach space of continuous functions $\mathfrak{u}:J^{\prime }\rightarrow \mathbb{R}$ with the norm $\Vert \mathfrak{u} \Vert =\max \{ \vert \mathfrak{u}(\sigma) \vert :\sigma \in J\}$. Clearly, $\mathcal{\mathbb{R} }$ is a Banach space with this norm, and hence the product space $\mathcal{\mathbb{R} }\times \mathcal{\mathbb{R} }$ is also a Banach space with the norm

$$\bigl\Vert (\mathfrak{u},\vartheta) \bigr\Vert = \Vert \mathfrak{u} \Vert + \Vert \vartheta \Vert .$$

We define the space $\mathcal{PC} (J)$ of piecewise continuous functions $\mathfrak{u}:J^{\prime }\rightarrow \mathbb{R}$ by

$$\mathcal{PC} (J) =\left \{ \textstyle\begin{array}{c} \mathfrak{u}:J^{\prime }\rightarrow \mathbb{R} ;\mathfrak{u}(\sigma)\in \mathcal{C} ((\sigma _{k},\sigma _{k+1}] ,\mathbb{R}) ;k=0,1,\ldots,m, \\ \mathfrak{u}(\sigma _{k}^{+})\text{ and }\mathfrak{u}(\sigma _{k}^{-})\text{ exist}\text{ }\text{ with }\mathfrak{u}(\sigma _{k}^{+})= \mathfrak{u}(\sigma _{k}^{-})\text{ for }k=0,1,\ldots,m\end{array}\displaystyle \right \} .$$

Obviously, $\mathcal{PC} (J)$ is a Banach space endowed with the norm

$$\Vert \mathfrak{u} \Vert _{\mathcal{PC} (J) }= \underset{\sigma \in J}{\max } \bigl\vert \mathfrak{u}(\sigma) \bigr\vert .$$

Define the product space $\mathcal{B}=\mathcal{PC} (J) \times \mathcal{PC} (J)$ with the norm

$$\bigl\Vert (\mathfrak{u},\vartheta) \bigr\Vert _{ \mathcal{B}}= \Vert \mathfrak{u} \Vert _{\mathcal{PC} (J) }+ \Vert \vartheta \Vert _{\mathcal{PC} (J) }$$

for $(\mathfrak{u},\vartheta) \in \mathcal{B}$.

Definition 2.1

([36])

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Let $\mathfrak{y}>0$ and $f\in L_{1} (J)$. Then the generalized RL fractional integral of a function *f* of order $\mathfrak{y}$ with respect to φ is defined as

$$\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }f(\sigma)= \frac{1}{\Gamma (\mathfrak{y})} \int _{0}^{\sigma }\varphi ^{\prime }(s) \bigl(\varphi (\sigma)-\varphi (s) \bigr) ^{\mathfrak{y}-1}f(s)\,ds.$$

Definition 2.2

([41])
Let $n-1<\mathfrak{y}<n\in \mathbb{N}$, and let $f,\varphi \in \mathcal{PC}^{n} (J)$. Then the generalized Hilfer fractional derivative of a function *f* of order $\mathfrak{y}$ and type $0\leq \mathfrak{p}\leq 1$ with respect to φ is defined as

$$\begin{aligned} ^{H}\mathcal{D}_{0^{+}}^{\mathfrak{y},\mathfrak{p},\varphi }f(\sigma) =&\mathcal{I}_{0^{+}}^{\mathfrak{p}(n-\mathfrak{y});\varphi }f_{ \varphi }^{[n]}\mathcal{I}_{0^{+}}^{ (1-\mathfrak{p}) (n- \mathfrak{y}) ,\varphi }f(\sigma) \\ =&\mathcal{I}_{0^{+}}^{\mathfrak{p}(n-\mathfrak{y});\varphi }f_{ \varphi }^{[n]} \mathcal{I}_{0^{+}}^{n-\mathfrak{\gamma },\varphi }f(\sigma) \\ =&\mathcal{I}_{0^{+}}^{\mathfrak{p}(n-\mathfrak{y});\varphi } \mathcal{D}_{a^{+}}^{\mathfrak{\gamma };\varphi }f(\sigma),\quad \mathfrak{\gamma }=\mathfrak{y}+n \mathfrak{p}-\mathfrak{yp}, \end{aligned}$$

where

$$\mathcal{D}_{0^{+}}^{\mathfrak{\gamma};\varphi }f(\sigma)=f_{ \varphi }^{[n]} \mathcal{I}_{0^{+}}^{(1-\mathfrak{p})(n-\mathfrak{y}); \varphi }f(\sigma),\quad \text{and} \quad f_{\varphi }^{[n]}= \biggl(\frac{1}{\varphi ^{\prime }(\sigma)} \frac{d}{d\sigma } \biggr) ^{n}.$$

Lemma 2.3

([41])

Let $\mathfrak{\gamma}=\mathfrak{y}+\mathfrak{p}-\mathfrak{yp}$, $\mathfrak{y}>0$, $\mathfrak{p}>0$, *and* $u\in \mathcal{PC}_{1-\mathfrak{\gamma };\varphi }^{\mathfrak{\gamma }} (J)$. *Then*

$$\mathcal{I}_{0^{+}}^{\mathfrak{\gamma};\varphi }\mathcal{D}_{0^{+}}^{\mathfrak{\gamma};\varphi }u= \mathcal{I}_{0^{+}}^{\mathfrak{y}; \varphi }\textit{ }^{H} \mathcal{D}_{0^{+}}^{\mathfrak{y},\mathfrak{p};\varphi }u\quad \textit{and} \quad \mathcal{D}_{0^{+}}^{\mathfrak{\gamma};\varphi } \mathcal{I}_{0^{+}}^{\mathfrak{y};\varphi }u= \mathcal{D}_{0^{+}}^{\mathfrak{p}(1-\mathfrak{y});\varphi }u.$$

Theorem 2.4

([41])

Let $0\leq \mathfrak{\gamma }<\mathfrak{y}$ *and* $u\in \mathcal{PC} (J)$. *Then* $\mathcal{I}_{0^{+}}^{\mathfrak{y};\varphi }\$ [illegible] $\lim I_{0^{+}}^{\mathfrak{y};\varphi }u(\sigma)=0.$

Lemma 2.5

([36, 41])

Let $\mathfrak{y},\mathfrak{p}>0$ *and* $\delta >0$. *Then*

$$\begin{aligned}& \mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\mathcal{I}_{0^{+}}^{ \mathfrak{p},\varphi }f(\sigma)=\mathcal{I}_{0^{+}}^{\mathfrak{y}+\mathfrak{p}, \varphi }f(\sigma), \\& \mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi } \bigl(\varphi (\sigma)- \varphi (0) \bigr) ^{\delta -1}= \frac{\Gamma (\mathfrak{\gamma })}{\Gamma (\mathfrak{y}+\mathfrak{\gamma })} \bigl(\varphi (\sigma)- \varphi (0) \bigr) ^{\mathfrak{y}+\delta -1}, \end{aligned}$$

and $^{H}\mathcal{D}_{0^{+}}^{\mathfrak{y},\mathfrak{p},\varphi } \bigl(\varphi (\sigma)-\varphi (0) \bigr) ^{\mathfrak{\gamma }-1}=0, \quad \mathfrak{\gamma }=\mathfrak{y}+n\mathfrak{p}-\mathfrak{yp}.$

Lemma 2.6

([41])

If $f\in \mathcal{PC}^{n} (J) , n-1<\mathfrak{y} <n,$ *and* $0\leq \mathfrak{p}\leq 1,$ *then*

\mathcal{I}_{0^{+}}^{\mathfrak{y};\varphi }\textit{ }^{H} \mathcal{D}_{0^{+}}^{\mathfrak{y},\mathfrak{p},\varphi }f(\sigma)=f(\sigma)-\sum _{k=1}^{n} \frac{ (\varphi (\sigma)-\varphi (0)) ^{\mathfrak{\gamma }-k}}{\Gamma [illegible] n-k] }

$\mathcal{I}_{a^{+}}^{(1-\mathfrak{p})(n-\mathfrak{y});\varphi }f(0)$,

and $^{H}\mathcal{D}_{0^{+}}^{\mathfrak{y},\mathfrak{p},\varphi } \mathcal{I}_{0^{+}}^{\mathfrak{y};\varphi }f(\sigma)=f(\sigma)$.

Lemma 2.7

([31] (Leray–Schauder alternative)) *Let* $\Xi :\mathcal{X}\rightarrow \mathcal{X}$ *be a completely continuous operator*, *and let* $\digamma (\Xi)= \{ y\in \mathcal{X}:y=\xi \Xi (y),\xi \in [0,1] \}$. *Then either the set* $\digamma (\Xi)$ *is unbounded*, *or* Ξ *has at least one fixed point*.

Theorem 2.8

([29] (Banach fixed point theorem)) *Let* $\mathcal{X}$ *be a Banach space*, *let* $K\subset \mathcal{X}$ *be closed*, *and let* $\Xi :K\rightarrow K$ *be a strict contraction*, *that is*, $\Vert \Xi (x)-\Xi (y) \Vert \leq L \Vert x-y \Vert$ *for some* $0< L<1$ *and all* $x,y\in K$. *Then* Ξ *has a fixed point in K*.

Lemma 2.9

Let $\mathfrak{\gamma }=\mathfrak{y}+\mathfrak{p}-\mathfrak{yp}$, $\mathfrak{y}\in (0,1)$, $\mathfrak{p}\in [0,1]$, *and let* $\varpi :J^{\prime }\rightarrow \mathbb{R}$ *be a continuous function*. *Then* $\mathfrak{u}\in \mathcal{PC}^{\mathfrak{\gamma }} (J)$ *satisfies*

\textstyle\begin{cases} \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \mathfrak{u}(\sigma)=\varpi (\sigma),&\sigma \in J:= [0,T] ,\sigma \neq \sigma _{k},k=1,\ldots,m, \\ \Delta

Z_{k}

\mathfrak{u}(\sigma _{k}^{-}),&k=1,\ldots,m,\\ \mathfrak{u}(T)=w \end{cases}

(2.1)

if and only if $\mathfrak{u}$ *satisfies the following integral equations*:

$$\mathfrak{u}(\sigma)= \textstyle\begin{cases} \frac{ (\varphi (\sigma)-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} [w-\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (T)] +\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma) ,\quad \sigma \in [0,\sigma _{1}] , \\ \sum_{i=1}^{k+1} \frac{ (\varphi (\sigma _{i})-\varphi (\sigma _{i-1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{i-1})) ^{\mathfrak{\gamma }-1}} [w-\mathcal{I}_{i-1^{+}}^{\mathfrak{y},\varphi }\varpi (s) (T)] +\sum_{i=1}^{k}\mathcal{I}_{\sigma _{i-1}^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{i}) \\ \quad {}+\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma) +\sum_{i=1}^{k}Z_{i}\mathfrak{u}(\sigma _{i}^{-}), \quad \sigma \in (\sigma _{k},\sigma _{k+1}] , k=1,\ldots,m.\end{cases}$$

(2.2)

Proof

First, let $\mathfrak{u}\in \mathcal{PC}^{\mathfrak{\gamma }} (J)$ be a solution of problem (2.1). We prove that $\mathfrak{u}$ is a solution of (2.2).

If $\sigma \in [0,\sigma _{1}]$, then $\mathcal{D}_{[\sigma] }^{\mathfrak{y,p,\varphi }} \mathfrak{u}(\sigma)=\varpi (\sigma), [\sigma] =0$. Taking the operator $\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }$ on both sides of the first equation in (2.1) and using Lemma 2.6, we have

$$\mathfrak{u}(\sigma)= \frac{ (\varphi (\sigma)-\varphi (0)) ^{\mathfrak{\gamma }-1}}{\Gamma (\mathfrak{\gamma })}\mathcal{I}_{0^{+}}^{1-\mathfrak{\gamma },\varphi } \mathfrak{u}(0)+\mathcal{I}_{0^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma) .$$

(2.3)

By the terminal condition we have

$$\mathcal{I}_{0^{+}}^{1-\mathfrak{\gamma },\varphi }\mathfrak{u}(0)= \frac{\Gamma (\mathfrak{\gamma })}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] .$$

(2.4)

Putting (2.4) into (2.3), we get

$$\mathfrak{u}(\sigma)= \frac{ (\varphi (\sigma)-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] +\mathcal{I}_{0^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma) .$$

This means

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$$\mathfrak{u}\bigl(\sigma _{1}^{-}\bigr)= \frac{ (\varphi (\sigma _{1})-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] +\mathcal{I}_{0^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) .$$

Since $\mathfrak{u}(\sigma _{1}^{-})=\mathfrak{u}(\sigma _{1}^{+})-Z_{1}\mathfrak{u}(\sigma _{1}^{-})$, we get

$$\mathfrak{u}\bigl(\sigma _{1}^{+}\bigr)= \frac{ (\varphi (\sigma _{1})-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] +\mathcal{I}_{0^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-}\bigr).$$

If $\sigma \in (\sigma _{1},\sigma _{2}]$, then $\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma)=\varpi (\sigma)$, $[\sigma] =\sigma _{1}$, and $\mathfrak{u}(\sigma)$ is given by

\begin{aligned} \mathfrak{u}(\sigma) =&\mathfrak{u}\bigl(\sigma _{1}^{+} \bigr)+ \frac{ (\varphi (\sigma)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma) \\ =&\frac{ (\varphi (\sigma _{1})-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi

(T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] + \frac{ (\varphi (\sigma)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma) +Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-} \bigr). \end{aligned}

This means that

\begin{aligned} \mathfrak{u}\bigl(\sigma _{2}^{-}\bigr) =& \frac{ (\varphi (\sigma _{1})-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] \\ &{}+ \frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma _{2}) +Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-}\bigr). \end{aligned}

Since \mathfrak{u}(\sigma _{2}^{-})=\mathfrak{u}(\sigma _{2}^{+})-Z_{2}\mathfrak{u}(\sigma _{2}^{-}), we get

$$\begin{aligned} \mathfrak{u}\bigl(\sigma _{2}^{+}\bigr) =& \frac{ (\varphi (\sigma _{1})-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] \\ &{}+ \frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma _{2}) +Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-}\bigr)+Z_{2}\mathfrak{u}\bigl(\sigma _{2}^{-}\bigr). \end{aligned}$$

If $\sigma \in (\sigma _{2},\sigma _{3}]$, then $\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma)=\varpi (\sigma)$, $[\sigma] =\sigma _{2}$, and $\mathfrak{u}(\sigma)$ is given by

$$\begin{aligned} \mathfrak{u}(\sigma) =&\mathfrak{u}\bigl(\sigma _{2}^{+} \bigr)+ \frac{ (\varphi (\sigma)-\varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{2}^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] +\mathcal{I}_{\sigma _{2}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma) \\ =&\frac{ (\varphi (\sigma _{1})-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }$$

\varpi (s) (T) \bigr] \\ &{}+ \frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+ \frac{ (\varphi (\sigma)-\varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{2})) ^{1\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{2}^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma _{2}) +\mathcal{I}_{\sigma _{2}^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma) +Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-} \bigr)+Z_{2}\mathfrak{u}\bigl(\sigma _{2}^{-} \bigr). \end{aligned}

This means that

\begin{aligned} \mathfrak{u}\bigl(\sigma _{3}^{-}\bigr) =& \frac{ (\varphi (\sigma _{1})- \varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] \\ &{}+ \frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+ \frac{ (\varphi (\sigma _{3})-\varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)- [illegible] a }-1}}

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\bigl[w-\mathcal{I}_{\sigma _{2}^{+}}^{

\mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma _{2}) +\mathcal{I}_{\sigma _{2}^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma _{3}) \\ &{}+Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-} \bigr)+Z_{2}\mathfrak{u}\bigl(\sigma _{2}^{-} \bigr). \end{aligned}

After impulse (\mathfrak{u}(\sigma _{3}^{-})=\mathfrak{u}(\sigma _{3}^{+})-Z_{3} \mathfrak{u}(\sigma _{3}^{-})) , we get \begin{aligned} \mathfrak{u}\bigl(\sigma _{3}^{-}\bigr) =& \frac{ (\varphi (\sigma _{1})- \varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] \\ &{}+ \frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+ \frac{ (\varphi (\sigma _{3})-\varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)- \varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{2}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma _{2}) +\mathcal{I}_{\sigma _{2}^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma _{3}) \\ &{}+Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-}

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\bigr)+Z_{2}\mathfrak{u}\bigl(\sigma _{2}^{-}

\bigr)+Z_{3} \mathfrak{u}\bigl(\sigma _{3}^{-} \bigr). \end{aligned}

If \sigma \in (\sigma _{3},\sigma _{4}] , then \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma)=\varpi (\sigma), [\sigma] =\sigma _{3}, and \mathfrak{u}(\sigma) is given by

\begin{aligned} \mathfrak{u}(\sigma) =&\mathfrak{u}\bigl(\sigma _{3}^{+} \bigr)+ \frac{ (\varphi (\sigma)-\varphi (\sigma _{3})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{3})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{3}^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] +\mathcal{I}_{\sigma _{3}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma) \\ =&\frac{ (\varphi (\sigma _{1})-\varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ &{}+ \frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+ \frac{ (\varphi (\sigma _{3})-\varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{2})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{2}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+ \frac{ (\varphi (\sigma)-\varphi (\sigma _{3})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{3})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{3}^{+}}^{\mathfrak{y},

\varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma _{2}) +\mathcal{I}_{\sigma _{2}^{+}}^{ \mathfrak{y},\varphi }\varpi (s) (\sigma _{3}) +\mathcal{I}_{\sigma _{3}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (\sigma) \\ & {}+Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-} \bigr)+Z_{2}\mathfrak{u}\bigl(\sigma _{2}^{-} \bigr)+Z_{3} \mathfrak{u}\bigl(\sigma _{3}^{-} \bigr). \end{aligned}

Assume that

\begin{aligned} \mathfrak{u}\bigl(\sigma _{k}^{+}\bigr) =& \frac{ (\varphi (\sigma _{1})- \varphi (0)) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (0)) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{0^{+}}^{\mathfrak{y}, \varphi } \varpi (s) (T) \bigr] \\ &{}+ \frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+\cdots + \frac{ (\varphi (\sigma _{k})-\varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} \bigl[w- \mathcal{I}_{\sigma _{k-1}^{+}}^{ \mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ & {}+\mathcal{I}_{0^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma _{1}) +\mathcal{I}_{\sigma _{1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (a _{k- 1}^{+}}^{\mathfrak{y},\varphi } \varpi (s) (

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\sigma _{k}) \\ & {}+Z_{1}\mathfrak{u}\bigl(\sigma _{1}^{-} \bigr)+Z_{2}\mathfrak{u}\bigl(\sigma _{2}^{-} \bigr)+ \cdots +Z_{k}\mathfrak{u}\bigl(\sigma _{k}^{-} \bigr). \end{aligned}

Then, inductively, for \sigma \in (\sigma _{k},\sigma _{k+1}] , we have \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\mathfrak{u} (\sigma)=\varpi (\sigma), [\sigma] =\sigma _{k} , and \mathfrak{u}(\sigma) is given by \begin{aligned} \mathfrak{u}(\sigma) =&\mathfrak{u}\bigl(\sigma _{k}^{+} \bigr)+ \frac{ (\varphi (\sigma)-\varphi (\sigma _{k})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{k})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] +\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma) \\ =&\sum_{i=1}^{k+1} \frac{ (\varphi (\sigma _{i})-\varphi (\sigma _{i-1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{i-1})) ^{\mathfrak{\gamma }-1}} \bigl[w-\mathcal{I}_{i-1^{+}}^{\mathfrak{y},\varphi } \varpi (s) (T) \bigr] \\ &{}+\sum_{i=1}^{k} \mathcal{I}_{\sigma _{i-1}^{+}}^{\mathfrak{y}, \varphi }\varpi (s) (\sigma _{i}) +\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }\varpi (s) (\sigma) +\sum_{i=1}^{k}Z_{i}\mathfrak{u}\bigl(\sigma _{i}^{-}\bigr). \end{aligned}

Thus (2.2) is satisfied

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Conversely, assume that $\mathfrak{u}$ satisfies equation (2.2).

Case 1: $\sigma \in [0,\sigma _{1}]$.

Replacing σ by T in (2.2), we get $\mathfrak{u}(T)=w$. On the other hand, applying $\mathcal{D}_{0^{+}}^{\mathfrak{\gamma };\varphi }$ to both sides of (2.2) and using Lemma 2.3, we get $\mathcal{D}_{0^{+}}^{\mathfrak{\gamma };\varphi }\mathfrak{u}(\sigma)=\mathcal{D}_{0^{+}}^{\mathfrak{p}(1-\mathfrak{y});\varphi }\varpi (\sigma)$.

(2.5)

Since $\mathfrak{u}\in \mathcal{PC}^{\mathfrak{\gamma }} (J)$, by definition of $\mathcal{PC}^{\mathfrak{\gamma }} (J)$ we have $\mathcal{D}_{0^{+}}^{\mathfrak{\gamma };\varphi }\mathfrak{u}\in \mathcal{PC} (J)$. So, (2.5) implies $\mathcal{D}_{0^{+}}^{\mathfrak{\gamma };\varphi }\mathfrak{u}(\sigma)=\mathcal{D}_{0^{+}}^{\mathfrak{p}(1-\mathfrak{y});\varphi }\varpi (\sigma)\in \mathcal{PC} (J)$.

For $\varpi \in \mathcal{PC} (J)$, it is obvious that $\mathcal{I}_{0^{+}}^{1-\mathfrak{p}(1-\mathfrak{y});\varphi }\varpi \in \mathcal{PC}^{1} (J)$. Hence ϖ and $\mathcal{I}_{0^{+}}^{1-\mathfrak{p}(1-\mathfrak{y});\varphi }\varpi$ satisfy the conditions of Theorem 2.6. Now, applying

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$\mathcal{I}_{0^{+}}^{\mathfrak{p}(1-$

$\mathfrak{y});\varphi }$ to both sides of (2.5) and using Theorem 2.6, we get

$$^{H}\mathcal{D}_{0^{+}}^{\mathfrak{y},\mathfrak{p};\varphi } \mathfrak{u}(\sigma)=\varpi (\sigma)-\frac{\mathcal{I}_{0^{+}}^{1-\mathfrak{p}(1-\mathfrak{y});\varphi }\varpi (0)}{\Gamma (\mathfrak{p}(1-\mathfrak{y}))}\bigl(\varphi (\sigma)-\varphi (0)\bigr)^{\mathfrak{p}(1-\mathfrak{y})-1}. \tag{2.6}$$

By Theorem 2.4 we have $\mathcal{I}_{0^{+}}^{1-\mathfrak{p}(1-\mathfrak{y});\varphi }\varpi (0)=0$. Hence (2.6) becomes

$$^{H}\mathcal{D}_{0^{+}}^{\mathfrak{y},\mathfrak{p};\varphi } \mathfrak{u}(\sigma)=\varpi (\sigma),\quad \sigma \in J.$$

Case 2: $\sigma \in (\sigma _{k},\sigma _{k+1}]$. By the same technique as in case 1 we can easily prove case 2. □

Lemma 2.10

Let $\mathfrak{\gamma }=\mathfrak{y}+\mathfrak{p}-\mathfrak{yp}$ *be such that* $\mathfrak{y}\in (0,1)$, $\mathfrak{p}\in [0,1]$, *and let* $f,g:J^{\prime }\times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ *be continuous functions. If* $(\mathfrak{u},\vartheta) \in \mathcal{B}$ *satisfies problem* (1.1), *then by Lemma* 2.9, $(\mathfrak{u},\vartheta)$ *satisfies the following integral equations*:

\textstyle\begin{cases} \mathfrak{u}(\sigma)=
\textstyle\begin{cases} \sum _{0< \sigma _{k}<
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\sigma } \frac{ (\varphi (\sigma _{k}) - \varphi

(\sigma _{k-1})) ^{\mathfrak{\gamma }-1}}{ (
\varphi (T)-\varphi (\sigma _{k-1}))
^{\mathfrak{\gamma }-1}} [w_{1}-
\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi }f(s,\mathfrak{u}
(s), \mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\vartheta (s)) (T)] \\
\quad {}+\sum_{0< \sigma _{k}< \sigma
}\mathcal{I}_{\sigma _{k-1}^{+}}^{
\mathfrak{y},\varphi }f(s,\mathfrak{u}
(s),\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\vartheta (s)) (\sigma
_{k}) + \mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi
}f(s,\mathfrak{u}(s), \mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\vartheta (s)) (\sigma
) \\ \quad {}+\sum_{0< \sigma _{k}< \sigma
}Z_{k}\mathfrak{u}(\sigma _{k}^{-}),\quad
\sigma \in (\sigma _{k},\sigma _{k+1}]
,k=1,\ldots,m,\end{cases}\displaystyle \\ \vartheta
(\sigma)= \textstyle\begin{cases} \sum_{0<
\sigma _{k}< \sigma } \frac{ (\varphi (\sigma
_{k})-\varphi (\sigma _{k-1}))
^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi
(\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} [
w_{2}-\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi
}g(s,\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\mathfrak{u}
(s),\vartheta (s)) (T)] \\ \quad {}+\sum_{0<
\sigma _{k}< \sigma }\mathcal{I}_{\sigma _{k-
1}^{+}}^{ \mathfrak{y},\varphi
}g(s,\mathcal{D}_{ [\sigma] }^{
\mathfrak{y,p,\varphi }}\mathfrak{u}(s),\vartheta

_{k}^{+}}^{\mathfrak{y},\varphi
}g(s,\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\mathfrak{u}
(s),\vartheta (s)) (\sigma) \\ \quad {}+\sum_{0<
\sigma _{k}< \sigma }Z_{k}\mathfrak{u}(\sigma
_{k}^{-}), \quad \sigma \in (\sigma _{k},\sigma
_{k+1}] ,k=1,\ldots,m.\end{cases}\displaystyle
\end{cases}

Consider the continuous operator \Xi
:\mathcal{B}\rightarrow \mathcal{B} defined by
\Xi (\mathfrak{u},\vartheta) (\sigma)= \bigl(\Xi
_{1} (\mathfrak{u},\vartheta) (\sigma),\Xi _{2} (
\vartheta , \mathfrak{u}) (\sigma) \bigr) ,

(2.7)

where
\Xi _{1} (\mathfrak{u},\vartheta) (\sigma)=
\textstyle\begin{cases} \sum_{0< \sigma _{k}<
\sigma } \frac{ (\varphi (\sigma _{k})-\varphi
(\sigma _{k-1})) ^{\mathfrak{\gamma }-1}}{ (
\varphi (T)-\varphi (\sigma _{k-1}))
^{\mathfrak{\gamma }-1}} [w_{1}-
\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi }f(s,\mathfrak{u}
(s), \mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\vartheta (s)) (T)] \\
\quad {}+\sum_{0< \sigma _{k}< \sigma
}\mathcal{I}_{\sigma _{k-1}^{+}}^{
\mathfrak{y},\varphi }f(s,\mathfrak{u}
(s),\mathcal{D}_{0}^{ \mathfrak{y,p,\varphi
}}\vartheta (s)) (\sigma _{k})\\ \quad {}
+\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi
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}f(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]

}^{\mathfrak{y,p,\varphi }}\vartheta (s)) (\sigma) \\ \quad {}+\sum_{0< \sigma _{k}< \sigma }Z_{k}\mathfrak{u}(\sigma _{k}^{-}), \quad \sigma \in (\sigma _{k},\sigma _{k+1}] , k=1,\ldots,m,\end{cases}

(2.8)

and
\Xi _{2} (\vartheta ,\mathfrak{u}) (\sigma)= \textstyle\begin{cases} \sum_{0< \sigma _{k}< \sigma } \frac{ (\varphi (\sigma _{k})-\varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} [w_{2}- \mathcal{I}_{\sigma _{k-1}^{+}}^{\mathfrak{y},\varphi }g(s,\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\mathfrak{u} (s),\vartheta (s)) (T)] \\ \quad {}+\sum_{0< \sigma _{k}< \sigma }\mathcal{I}_{\sigma _{k-1}^{+}}^{ \mathfrak{y},\varphi }g(s,\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\mathfrak{u}(s),\vartheta (s)) (\sigma _{k}) \\ \quad {}+ \mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }g(s,\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\mathfrak{u} (s),\vartheta (s)) (\sigma) \\ \quad +\sum_{0< \sigma _{k}< \sigma }Z_{k}\mathfrak{u}(\sigma _{k}^{-}), \quad \sigma \in (\sigma _{k},\sigma _{k+1}] , k=1,\ldots,m.\end{cases}

(2.9)

Note that the fixed points of the operator Ξ are solutions of problem (1.1).

3 Existence of solution

In this section, we consider a general coupled system of Hilfer FDEs (1.1) involving an arbitrary function φ. To demonstrate our main results, we introduce the following hypotheses.

(H_1) The functions $f,g:J\times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and there exist constant numbers $\varrho _{f},\varrho _{g},\varrho _{f}^{\prime },\varrho _{g}^{ \prime }>0$ such that for all $(\mathfrak{u},\vartheta) , (\widehat{\mathfrak{u}},\widehat{\vartheta }) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} \bigl\vert f\bigl(\sigma ,\mathfrak{u} (\sigma) ,\vartheta (\sigma) \bigr)-f\bigl(\sigma ,\widehat{\mathfrak{u}} (\sigma) ,\widehat{ \vartheta } (\sigma) \bigr) \bigr\vert \leq &\varrho _{f} \bigl\vert \mathfrak{u} (\sigma) - \widehat{\mathfrak{u}} (\sigma) \bigr\vert +\varrho _{f}^{\prime } \bigl\vert \vartheta (\sigma) -\widehat{\vartheta } (\sigma) \bigr\vert , \\ \bigl\vert g\bigl(\sigma ,\mathfrak{u} (\sigma) ,\vartheta (\sigma) \bigr)-g\bigl(\sigma ,\widehat{\mathfrak{u}} (\sigma) ,\widehat{ \vartheta } (\sigma) \bigr) \bigr\vert \leq &\varrho _{g} \bigl\vert \mathfrak{u} (\sigma) - \widehat{\mathfrak{u}} (\sigma) \bigr\vert +\varrho _{g}^{\prime } \bigl\vert \vartheta (\sigma) -\widehat{\vartheta } (\sigma) \bigr\vert . \end{aligned}$$

(H_2) The functions $f,g:J\times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are

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continuous functions such that for each $(\mathfrak{u},\vartheta) \in \mathbb{R}$, there exist nondecreasing continuous linear functions $\omega _{f},\omega _{g}:\mathbb{R}^{+}\rightarrow \mathbb{R}^{+}$ such that

$$\begin{aligned} \bigl\vert f\bigl(\sigma ,\mathfrak{u} (\sigma) ,\vartheta (\sigma) \bigr) \bigr\vert \leq &\omega _{f} \bigl\vert \mathfrak{u} (\sigma) \bigr\vert +\omega _{f}^{ \prime } \bigl\vert \vartheta (\sigma) \bigr\vert , \\ \bigl\vert g\bigl(\sigma ,\mathfrak{u} (\sigma) ,\vartheta (\sigma) \bigr) \bigr\vert \leq &\omega _{g} \bigl\vert \mathfrak{u} (\sigma) \bigr\vert +\omega _{g}^{ \prime } \bigl\vert \vartheta (\sigma) \bigr\vert . \end{aligned}$$

(H_3) The functions $Z_{k}:\mathbb{R} \rightarrow \mathbb{R}$ are continuous, and there exists a constant $L_{Z}>0$ such that $\bigl\vert Z_{k}(\Theta)-Z_{k}\bigl(\Theta ^{\ast }\bigr) \bigr\vert \leq L_{Z} \bigl\vert \Theta - \Theta ^{\ast } \bigr\vert ,\quad k=1, \ldots,m, \Theta ,\Theta ^{\ast }\in \mathbb{R}$.

In the following, we will apply the Theorem 2.7 to obtain an existence result for system (1.1).

Theorem 3.1

Assume that (H_1)–(H_3) *hold. If* $Q_{1}:= \frac{ (2m+1) [\omega _{f} (1+\omega _{g}) +\omega _{g}^{\prime }(1+\omega _{f}^{\prime })] }{2\Gamma (\mathfrak{y}+1) (1-\omega _{f}^{\prime }\omega _{g}) } \bigl(\varphi (T)-\varphi (0) \bigr)$

then problem (1.1) has at least one solution on J.

Proof

Define the closed ball set

$$\mathbb{B}_{R}= \biggl\{ (\mathfrak{u},\vartheta) \in \mathcal{B}: \bigl\Vert (\mathfrak{u},\vartheta) \bigr\Vert _{ \mathcal{PC} (J) }\leq R,\text{ } \Vert \mathfrak{u} \Vert _{\mathcal{PC} (J) }\leq \frac{R}{2},\text{ } \Vert \vartheta \Vert _{\mathcal{PC} (J) }\leq \frac{R}{2} \biggr\}$$

with

$$R\geq \frac{m [\vert w_{1} \vert + \vert w_{2} \vert] }{1-Q_{1}}.$$

We will prove that the operator Ξ defined by (2.7) has a fixed point by using Theorem 2.7. For this, we divide the proof into three steps.

Step 1: $\Xi (\mathbb{B}_{R})\subset \mathbb{B}_{R}$.

For any $(\mathfrak{u},\vartheta) \in \mathbb{B}_{R}$, we have

$$\bigl\Vert \Xi (\mathfrak{u},\vartheta) \bigr\Vert _{ \mathcal{B}}\leq \bigl\Vert \Xi _{1} (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{PC} (J) }+ \bigl\Vert \Xi _{2} (\vartheta ,\mathfrak{u}) \bigr\Vert _{\mathcal{PC} (J) }.$$

From equation (2.8) we have

$$\begin{aligned} \bigl\vert \Xi _{1} (\mathfrak{u},\vartheta) \bigr\vert \leq &\sum_{0< \sigma _{k}< \sigma } \frac{ (\varphi (\sigma _{k})-\varphi (\sigma _{k-1})) \ldots \varphi$$

(\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} \\ &
\bigl[\vert w_{1} \vert +\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert
f\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }} \vartheta (s)\bigr)
\bigr\vert (T) \bigr] \\ &{}+\sum_{0< \sigma
_{k}< \sigma }\mathcal{I}_{\sigma _{k-1}^{+}}^{
\mathfrak{y},\varphi } \bigl\vert
f\bigl(s,\mathfrak{u}(s),\mathcal{D}_{0}^{
\mathfrak{y,p,\varphi }} \vartheta (s)\bigr)
\bigr\vert (\sigma _{k}) \\ &
{}+\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert f
\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{ \mathfrak{y,p,\varphi }} \vartheta (s)\bigr)
\bigr\vert (\sigma) \\ &{}+\sum_{0< \sigma _{k}
< \sigma } \bigl\vert Z_{k}
\mathfrak{u}\bigl(\sigma _{k}^{-}\bigr)
\bigr\vert \\ \leq &m \vert w_{1} \vert
+m\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert
f\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }} \vartheta (s)\bigr)
\bigr\vert (T) \\ &{}+m\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert f
\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{ \mathfrak{y,p,\varphi }} \vartheta (s)\bigr)
\bigr\vert (\sigma _{k}) \\ &
{}+\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert f
\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{ \mathfrak{y,p,\varphi }} \vartheta (s)\bigr)
\bigr\vert (\sigma) +mL_{Z} \bigl\vert
\mathfrak{u}(\sigma) \bigr\vert \\ \leq &m \vert

 _{f}

\Vert \mathfrak{u} \Vert _{\mathcal{PC} (J) }+\omega _{f}^{\prime }\omega _{g}^{\prime } \Vert \vartheta \Vert _{\mathcal{PC} (J) }} {\Gamma (\mathfrak{y}+1) (1-\omega _{f}^{\prime }\omega _{g}) } \biggr] \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}} \\ & {}+mL_{Z} \Vert \mathfrak{u} \Vert _{\mathcal{PC} (J) } \\ \leq &m \vert w_{1} \vert + (2m+1) \biggl[\frac{ (\omega _{f}+\omega _{f}^{\prime }\omega _{g}^{\prime }) R} {2\Gamma (\mathfrak{y}+1) (1-\omega _{f}^{\prime }\omega _{g}) } \biggr] \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}}+mL_{Z} \frac{R}{2}. \end{aligned}

Using the same technique, we get \bigl\Vert \Xi _{2} (\vartheta ,\mathfrak{u}) \bigr\Vert _{\mathcal{PC} (J) }\leq m \vert w_{2} \vert + (2m+1) \biggl[\frac{ (\omega _{f}\omega _{g}+\omega _{g}^{\prime }) R} {2\Gamma (\mathfrak{y}+1) (1-\omega _{f}^{\prime }\omega _{g}) } \biggr] \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}}+mL_{Z}\frac{R}{2}.

Thus \begin{aligned} \bigl\Vert \Xi (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{B}} \leq & \bigl\Vert \Xi _{1} (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{PC} (J) }+ \bigl\Vert \Xi _{2} (\vartheta ,\mathfrak{u}) \bigr\Vert _{\mathcal{PC} (J) } \\ \leq &m \bigl[\vert w_{1} \vert + \vert w_{2} \vert \bigr] +O_{1}R\leq R. \end{aligned}

Hence \Xi (\mathbb{B}_{R})\subset \mathbb{B}_{R}.

Step 2: Ξ is continuous and compact.

Let (\mathfrak{u}_{n},\vartheta _{n}) be a sequence such that (\mathfrak{u}_{n},\vartheta _{n}) \rightarrow (\mathfrak{u},\vartheta) in \mathbb{B}_{R}. Then we have

\begin{aligned}& \bigl\vert \Xi _{1} (\mathfrak{u}_{n},\vartheta _{n}) (\sigma)-\Xi _{1} (\mathfrak{u},\vartheta) (\sigma) \bigr\vert \\& \quad \leq \sum_{0< \sigma _{k}< \sigma }\mathcal{I}_{\sigma _{k-1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert \bigl[f\bigl(s,\mathfrak{u}_{n}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta _{n}(s) \bigr)-f\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }} \vartheta (s)\bigr) \bigr] \bigr\vert (T) \\& \qquad {}+\sum_{0< \sigma _{k}< \sigma }\mathcal{I}_{\sigma _{k-1}^{+}}^{ \mathfrak{y},\varphi } \bigl\vert \bigl[f\bigl(s,\mathfrak{u}_{n}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta _{n}(s) \bigr)-f\bigl(s, \mathfrak{u}(s),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \vartheta (s)\bigr) \bigr] \bigr\vert (\sigma _{k}) \\& \qquad {}+\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert \bigl[f\bigl(s, \mathfrak{u}_{n}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }} \vartheta _{n}(s)\bigr)-f\bigl(s,\mathfrak{u}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta (s)\bigr) \bigr] \bigr\vert (\sigma) \\& \qquad {}+\sum_{0< \sigma _{k}< \sigma }

 ma

_{k}^{-} \bigr)-Z_{k}\mathfrak{u}\bigl(\sigma _{k}^{-} \bigr) \bigr\vert \\& \quad \leq m\mathcal{I}_{\sigma _{k-1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert \bigl[f \bigl(s,\mathfrak{u}_{n}(s),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \vartheta _{n}(s)\bigr)-f\bigl(s,\mathfrak{u}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta (s)\bigr) \bigr] \bigr\vert (T) \\& \qquad {}+m\mathcal{I}_{\sigma _{k-1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert \bigl[f\bigl(s, \mathfrak{u}_{n}(s),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \vartheta _{n}(s)\bigr)-f\bigl(s,\mathfrak{u}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta (s)\bigr) \bigr] \bigr\vert (\sigma _{k}) \\& \qquad {}+\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert \bigl[f\bigl(s, \mathfrak{u}_{n}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }} \vartheta _{n}(s)\bigr)-f\bigl(s,\mathfrak{u}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta (s)\bigr) \bigr] \bigr\vert (\sigma) \\& \qquad {}+mL_{Z} \bigl\vert \bigl[\mathfrak{u}_{n}(\sigma)-\mathfrak{u}(\sigma) \bigr] \bigr\vert \\& \quad \leq (2m+1) \frac{\varrho _{f} \Vert \mathfrak{u}_{n}-\mathfrak{u} \Vert _{\mathcal{PC} (J) }+\varrho _{f}^{\prime }\varrho _{g}^{\prime } \Vert \vartheta _{n}-\vartheta \Vert _{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y+}1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \Vert \mathfrak{u}_{n}-\mathfrak{u}

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By the same technique we get

$$\begin{aligned}& \bigl\vert \Xi _{2} (\mathfrak{u}_{n},\vartheta _{n}) (\sigma)-\Xi _{2} (\mathfrak{u},\vartheta) (\sigma) \bigr\vert \\& \quad \leq (2m+1) \frac{\varrho _{g}\varrho _{f} \Vert \mathfrak{u}_{n}-\mathfrak{u} \Vert _{\mathcal{PC} (J) }+\varrho _{g}^{\prime } \Vert \vartheta _{n}-\vartheta \Vert _{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y+}1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \Vert \vartheta _{n}-\vartheta \Vert _{\mathcal{PC} (J) }. \end{aligned}$$

Thus

\begin{aligned}& \bigl\Vert \Xi (\mathfrak{u}_{n},\vartheta _{n}) - \Xi (\mathfrak{u},\vartheta) \bigr\Vert _{ \mathcal{B}} \\& \quad \leq \bigl\Vert \Xi _{1} (\mathfrak{u}_{n}, \vartheta _{n}) -\Xi _{1} (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{PC} (J) }+ \bigl\Vert \Xi _{2} (\mathfrak{u}_{n},\vartheta _{n}) -\Xi _{2} (\mathfrak{u}, \vartheta) \bigr\Vert _{\mathcal{PC} (J) } \\& \quad \leq (2m+1) \biggl[\frac{ (\varrho _{f}+\varrho _{g}\varrho _{f}) \Vert \mathfrak{u}_{n}-\mathfrak{u} \Vert _{\mathcal{PC} (J) }+ (\varrho _{f}^{\prime }\varrho _{g}^{\prime }+\varrho _{g}^{\prime }) \Vert \vartheta _{n}-\vartheta \Vert _{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y+}1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \biggr] \bigl(\varphi (T)- \varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \bigl[\Vert \mathfrak{u}_{n}-\mathfrak{u} \Vert _{\mathcal{PC} (J) }+ \Vert \vartheta _{n}-

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$$\vartheta \Vert _{\mathcal{PC} (J) } \bigr] \\& \quad \rightarrow 0\quad \text{as } (\mathfrak{u}_{n},\vartheta _{n}) \rightarrow (\mathfrak{u},\vartheta) . \end{aligned}$$

Hence Ξ is continuous. Also, the operator Ξ is bounded on $\mathbb{B}_{R}$. Thus Ξ is uniformly bounded on $\mathbb{B}_{R}$. Next, we prove that Ξ is equicontinuous. Let $\sigma _{1},\sigma _{2}\in J$ be such that $\sigma _{1}<\sigma _{2}$. In view of (H_2), fixing $\sup_{ (\sigma , (\mathfrak{u},\vartheta)) \in J\times \mathbb{B}_{R}} \vert f(\sigma ,\mathfrak{u},\vartheta) \vert = \widehat{f}$ and $\sup_{ (\sigma , (\mathfrak{u},\vartheta)) \in J\times \mathbb{B}_{R}} \vert g(\sigma , \mathfrak{u},\vartheta) \vert =\widehat{g}$, we have

$$\begin{aligned}& \bigl\vert \Xi _{1} \bigl(\mathfrak{u}(\sigma _{2}),\vartheta (\sigma _{2}) \bigr) -\Xi _{1} \bigl(\mathfrak{u}(\sigma _{1}), \vartheta (\sigma _{1}) \bigr) \bigr\vert \\& \quad =\bigl\vert \mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }f \bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }} \vartheta (s)\bigr) (\sigma _{2}) \\& \qquad {} -\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }f\bigl(s, \mathfrak{u}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }} \vartheta (s)\bigr) (\sigma _{1}) \bigr\vert \\& \quad \leq \widehat{f} \biggl[\frac{ (\varphi (\sigma _{2})-\varphi (\sigma _{1})) ^{\mathfrak{y}}}{\Gamma (\mathfrak{y})}- \frac{ (\varphi (\sigma _{1})- [illegible] amma (\mathfrak{y})} \biggr]. \end{aligned}$$

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(3.1)

From (3.1) we have

$$\bigl\Vert \Xi _{1} \bigl(\mathfrak{u}(\sigma _{2}),\vartheta (\sigma _{2}) \bigr) -\Xi _{1} \bigl(\mathfrak{u}(\sigma _{1}), \vartheta (\sigma _{1}) \bigr) \bigr\Vert _{\mathcal{PC} (J) }\rightarrow 0\quad \text{as }\sigma _{2}\rightarrow \sigma _{1}.$$

(3.2)

By the same technique we get

$$\bigl\Vert \Xi _{2} \bigl(\mathfrak{u}(\sigma _{2}),\vartheta (\sigma _{2}) \bigr) -\Xi _{2} \bigl(\mathfrak{u}(\sigma _{1}), \vartheta (\sigma _{1}) \bigr) \bigr\Vert _{\mathcal{PC} (J) }\rightarrow 0\quad \text{as }\sigma _{2}\rightarrow \sigma _{1}.$$

(3.3a)

It follows from (3.2) and (3.3a) that

$$\bigl\Vert \Xi \bigl(\mathfrak{u}(\sigma _{2}),\vartheta (\sigma _{2}) \bigr) -\Xi \bigl(\mathfrak{u}(\sigma _{1}),\vartheta (\sigma _{1}) \bigr) \bigr\Vert _{{\mathcal{B}}}\rightarrow 0\quad \text{as }\sigma _{2} \rightarrow \sigma _{1}.$$

Hence Ξ is equicontinuous. By the Arzelà–Ascoli theorem we infer that Ξ is compact in $\mathbb{B}_{R}$. Therefore from the above steps we conclude that Ξ is completely continuous.

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Step 3: The set $\digamma = \{ (\mathfrak{u},\vartheta) \in \mathcal{B}: (\mathfrak{u},\vartheta) =\xi \Xi (\mathfrak{u},\vartheta) ,\text{ }\xi \in (0,1) \}$ is bounded.

Let $(\mathfrak{u},\vartheta) \in \digamma$. Then $(\mathfrak{u},\vartheta) =\xi \Xi (\mathfrak{u},\vartheta)$. Now, for $\sigma \in J$, we have $\mathfrak{u}(\sigma)=\xi \Xi _{1} (\mathfrak{u},\vartheta)$ and $\vartheta (\sigma)=\xi \Xi (\mathfrak{u},\vartheta)$. According to our hypotheses, we attain

$$\begin{aligned} \bigl\vert \mathfrak{u}(\sigma) \bigl(\varphi (\sigma)-\varphi (0) \bigr) ^{1-\mathfrak{\gamma }} \bigr\vert =& \bigl\vert \xi \Xi _{1} (\mathfrak{u},\vartheta) \bigr\vert \\ \leq & \bigl\Vert \Xi _{1} (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{PC} (J) }. \end{aligned}$$

By step 1 we have

$$\begin{aligned} \Vert \mathfrak{u} \Vert _{\mathcal{PC} (J) } =& \bigl\Vert \xi \Xi _{1} (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{PC} (J) } \\ \leq & \bigl\Vert \Xi _{1} (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{PC} (J) } \\ \leq &m \vert w_{1} \vert + (2m+1) \biggl[\frac{ (\omega _{f}+\omega _{f}^{\prime }\omega _{g}^{\prime }) R}{2\Gamma (\mathfrak{y}+1) (1-\omega _{f}^{\prime }\omega _{g}) } \biggr] \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}}+mL_{Z} \frac{R}{2} \end{aligned}$$

(3.4)

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and

$$\begin{aligned} & \\ \Vert \vartheta \Vert _{\mathcal{PC} (J) } \leq m \vert w_{2} \vert &+ (2m+1) \biggl[\frac{ (\omega _{f}\omega _{g}+\omega _{g}^{\prime }) R}{2\Gamma (\mathfrak{y}+1) (1-\omega _{f}^{\prime }\omega _{g}) } \biggr] \bigl(\varphi (T)- \varphi (0) \bigr) ^{\mathfrak{y}}+mL_{Z} \frac{R}{2}. \end{aligned}$$

(3.5)

From (3.4) and (3.5) we have

$$\bigl\Vert (\mathfrak{u},\vartheta) \bigr\Vert _{ \mathcal{B}}= \Vert \mathfrak{u} \Vert _{\mathcal{PC} (J) }+ \Vert \vartheta \Vert _{\mathcal{PC} (J) } \leq R.$$

Hence the set F is bounded. According to the above steps, together with Theorem 2.7, we conclude that Ξ has at least one fixed point. Consequently, system (1.1) has at least one solution on J. □

In the following theorem, we prove the uniqueness of solutions to system (1.1) by using Theorem 2.8.

Theorem 3.2

Assume that (H_1)–(H_3) *hold. If*

$$\mathcal{Q}= (2m+1) \rho +mL_{Z}< 1,$$

where $\rho =\max \{\rho _{1},\rho _{2}\}$ *with*

$$\begin{aligned}& \rho _{1}= \frac{\varrho _{f} (1+\varrho _{g}) }{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) }, \\& \rho _{2}= \frac{\varrho _{g}^{\prime }(1+\varrho _{f}^{\prime })}{\Gamma (\mathfrak{y}+1) (1-$$

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\varrho _{f}^{\prime }\varrho _{g}) },
\end{aligned}

then system (1.1) *has a unique solution.*

Proof

Consider the closed ball \mathbb{B}_{R} defined in Theorem 3.1. First, we show that \Xi (\mathbb{B}_{R})\subset \mathbb{B}_{R}. By the first step in Theorem 3.1 we have \Xi (\mathbb{B}_{R})\subset \mathbb{B}_{R}. Next, we need to prove that Ξ is a contraction map. Indeed, for (\mathfrak{u},\vartheta), (\widehat{\mathfrak{u}}, \widehat{\vartheta })\in \mathbb{B}_{R} and \sigma \in J, we obtain \begin{aligned}& \bigl\vert \bigl(\Xi _{1} (\mathfrak{u},\vartheta) (\sigma) -\Xi _{1} (\widehat{\mathfrak{u}}, \widehat{\vartheta }) (\sigma) \bigr) \bigr\vert \\& \quad \leq \sum_{0< \sigma _{k}< \sigma } \frac{ (\varphi (\sigma _{k})-\varphi (\sigma _{k-1})) ^{1-\mathfrak{\gamma }}}{ (\varphi (T)-\varphi (\sigma _{k-1})) ^{1-\mathfrak{\gamma }}} \\& \qquad {}\times \bigl[\mathcal{I}_{\sigma _{k-1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert f\bigl(s, \mathfrak{u}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\vartheta (s) \bigr)-f\bigl(s,\widehat{\mathfrak{u}}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(s)\bigr) \bigr\vert (T) \bigr] \\& \qquad {}+\sum_{0< \sigma _{k}< \sigma }\mathcal{I}_{\sigma _{k-1}^{+}}^{ \mathfrak{y},\varphi } \bigl\vert f\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \vartheta (s)\bigr)-

f\bigl(s, \widehat{\mathfrak{u}}(s),\mathcal{D}_{

[\sigma] }^{\mathfrak{y,p,\varphi }}
\widehat{\vartheta }(s) \bigr) \bigr\vert (\sigma
_{k}) \\& \qquad {}+\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert f
\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{ \mathfrak{y,p,\varphi }} \vartheta (s)\bigr)-
f\bigl(s,\widehat{\mathfrak{u}}(s),\mathcal{D}_{
[\sigma] }^{\mathfrak{y,p,\varphi }}
\widehat{\vartheta }(s)\bigr) \bigr\vert (\sigma)
\\& \qquad {}+\sum_{0< \sigma _{k}< \sigma }
\bigl\vert Z_{k} \mathfrak{u}\bigl(\sigma
_{k}^{-}\bigr)-Z_{k}
\widehat{\mathfrak{u}}\bigl(\sigma
_{k}^{-}\bigr) \bigr\vert \\& \quad \leq
m\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert \bigl[f
\bigl(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{ \mathfrak{y,p,\varphi }}\vartheta (s)\bigr)-
f\bigl(s,\widehat{\mathfrak{u}}(s), \mathcal{D}_{
[\sigma] }^{\mathfrak{y,p,\varphi }}
\widehat{\vartheta }(s)\bigr) \bigr] \bigr\vert (T)
\\& \qquad {}+m\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi } \bigl\vert \bigl[
f\bigl(s, \mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\vartheta (s) \bigr)-
f\bigl(s,\widehat{\mathfrak{u}}(s), \mathcal{D}_{
[\sigma] }^{\mathfrak{y,p,\varphi }}
\widehat{\vartheta }(s)\bigr) \bigr] \bigr\vert (
\sigma _{k}) \\& \qquad {}+\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert
\bigl[f\bigl(s, \mathfrak{u}(s),\mathcal{D}_{ [
\sigma] }^{ \mathfrak{y,p,\varphi }}\vartheta (s)
\bigr)-f\bigl(s,\widehat{\mathfrak{u}}(s),
\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi
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\sigma) \\& \qquad {}+m \bigl\vert Z_{k}\mathfrak{u}\bigl(\sigma _{k}^{-}\bigr)- Z_{k} \widehat{ \mathfrak{u}}\bigl(\sigma _{k}^{-}\bigr) \bigr\vert \\& \quad \leq (2m+1) \biggl[\frac{ (\varrho _{f} \Vert \mathfrak{u}- \widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }+\varrho _{f}^{\prime }\varrho _{g}^{\prime } \Vert \vartheta -\widehat{\vartheta } \Vert _{\mathcal{PC} (J) }) }{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \biggr] \bigl(\varphi (T)- \varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }, \end{aligned}

and, consequently, we obtain

\begin{aligned}& \bigl\Vert \Xi _{1} (\mathfrak{u},\vartheta) (\sigma) - \Xi _{1} (\widehat{\mathfrak{u}}, \widehat{\vartheta }) \bigr\Vert _{\mathcal{PC} (J) } \end{aligned}

(3.6)

\begin{aligned}& \quad \leq (2m+1) \biggl[\frac{\varrho _{f} \Vert \mathfrak{u}- \widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }+\varrho _{f}^{\prime }\varrho _{g}^{\prime } \Vert \vartheta -\widehat{\vartheta } \Vert _{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \biggr] \bigl(\varphi (T)- \varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }. \end{aligned}

By the same way we obtain

\begin{aligned}& \bigl\Vert \Xi _{2} (\mathfrak{u},\vartheta) (\sigma) - \Xi _{2} (\widehat{\mathfrak{u}}, \widehat{\vartheta }) \bigr\Vert _{\mathcal{PC} (J) } \\& \quad \leq (2m+1) \biggl[\frac{\varrho _{f}\varrho _{g} \Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }+\varrho _{g}^{\prime } \Vert \vartheta -\widehat{\vartheta } \Vert _{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \biggr] \bigl(\varphi (T)- \varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \Vert \vartheta -\widehat{\vartheta } \Vert _{ \mathcal{PC} (J) }. \end{aligned}

(3.8)

From (3.7) and (3.8) it follows that

\begin{aligned}& \bigl\Vert \Xi (\mathfrak{u},\vartheta) (\sigma) -\Xi (\widehat{ \mathfrak{u}}, \widehat{\vartheta }) \bigr\Vert _{\mathcal{B}} \\& \quad \leq \bigl\Vert \Xi _{1} (\mathfrak{u},\vartheta) (\sigma) -\Xi _{1} (\widehat{\mathfrak{u}}, \widehat{\vartheta }) \bigr\Vert _{\mathcal{PC} (J) }+ \bigl\Vert \Xi _{2} (\mathfrak{u},\vartheta) (\sigma) -\Xi _{2} (\widehat{\mathfrak{u}},\widehat{\vartheta }) \bigr\Vert _{\mathcal{PC} (J) } \\& \quad \leq (2m+1) \biggl[\frac{\varrho _{f} (1+\varrho _{g}) \Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) }+ \frac{\varrho _{g}^{\prime }(1+\varrho _{f}^{\prime }) \Vert \vartheta - \widehat{\vartheta } \Vert _{\mathcal{PC} (J) }}

{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \biggr] \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \bigl[\Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }+ \Vert \vartheta - \widehat{\vartheta } \Vert _{\mathcal{PC} (J) } \bigr] \\& \quad \leq (2m+1) \bigl[\rho _{1} \Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }+\rho _{2} \Vert \vartheta -\widehat{\vartheta } \Vert _{ \mathcal{PC} (J) } \bigr] \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}} \\& \qquad {}+mL_{Z} \bigl[\Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{\mathcal{PC} (J) }+ \Vert \vartheta - \widehat{\vartheta } \Vert _{\mathcal{PC} (J) } \bigr] \\& \quad \leq \bigl[(2m+1) \rho +mL_{Z} \bigr] \bigl[\Vert \mathfrak{u}-\widehat{\mathfrak{u}} \Vert _{ \mathcal{PC} (J) }+ \Vert \vartheta -\widehat{\vartheta } \Vert _{\mathcal{PC} (J) } \bigr] \\& \quad \leq \mathcal{Q} \bigl\Vert (\mathfrak{u},\vartheta) - (\widehat{ \mathfrak{u}},\widehat{\vartheta }) \bigr\Vert _{\mathcal{B}}. \end{aligned}

Thus the operator Ξ is a contraction. So by Theorem 2.8 system (1.1) has a unique solution. □

4 Stability analysis

To state the main theorem, we need the following definitions. Let \epsilon _{i}>0 and \lambda _{\phi _{i}}:J \rightarrow [0,\infty) (i=1,2) be continuous functions. We consider the following inequalities:

\begin{aligned} \bigl\vert \mathcal{D}_{ [\sigma

\mathfrak{u}}(\sigma)-f\bigl(\sigma ,\widehat{\mathfrak{u}}(\sigma),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(\sigma)\bigr) \bigr\vert \leq &\epsilon _{1}, \end{aligned}

(4.1)

$$\begin{aligned} \bigl\vert \mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{ \vartheta } (\sigma)-f\bigl(\sigma ,\mathcal{D}_{0}^{ \mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}} (\sigma),\widehat{\vartheta }(\sigma)\bigr) \bigr\vert \leq &\epsilon _{2} , \end{aligned}$$

(4.2)

$$\begin{aligned} \bigl\vert \mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{ \mathfrak{u}}(\sigma)-f\bigl(\sigma ,\widehat{\mathfrak{u}}(\sigma),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(\sigma)\bigr) \bigr\vert \leq &\epsilon _{1}\lambda _{\phi _{1}}(\sigma), \end{aligned}$$

(4.3)

$$\begin{aligned} \bigl\vert \mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{ \vartheta } (\sigma)-f\bigl(\sigma ,\mathcal{D}_{0}^{ \mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}} (\sigma),\widehat{\vartheta }(\sigma)\bigr) \bigr\vert \leq &\epsilon _{2}\lambda _{\phi _{2}}(\sigma). \end{aligned}$$

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(4.4)

Definition 4.1

([39])

System (1.1) is UH stable if there exists a real number $\mathcal{M}>0$ such that for each $\epsilon =\max \{\epsilon _{1},\epsilon _{2}\}>0$, there exists a solution $(\widehat{\mathfrak{u}},\widehat{\vartheta }) \in \mathcal{B}$ of inequalities (4.1) and (4.2) corresponding to a solution $(\mathfrak{u},\vartheta) \in \mathcal{B}$ of system (1.1) such that $\bigl\Vert (\widehat{\mathfrak{u}},\widehat{\vartheta }) - (\mathfrak{u}, \vartheta) \bigr\Vert _{ \mathcal{B}}\leq \mathcal{M}\epsilon,\quad \varkappa \in J.$

Definition 4.2

([39])

System (1.1) is UHR stable with respect to the nondecreasing function $\lambda _{\phi }(\sigma)=\max_{\varkappa \in J}\{\lambda _{\phi _{1}}(\sigma),\lambda _{\phi 2}(\sigma)\}$ if there exists a real number $\mathcal{N}>0$ such that for each solution $(\widehat{\mathfrak{u}},\widehat{\vartheta }) \in \mathcal{B}$ of inequalities (4.3) and (4.4), there exists a solution $(\mathfrak{u},\vartheta) \in \mathcal{B}$ of system (1.1) such that $\bigl\Vert (\widehat{\mathfrak{u}},\widehat{\vartheta }) - (\mathfrak{u}, \vartheta) \bigr\Vert _{ \mathcal{B}}\leq \mathcal{N}\epsilon \lambda$

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Remark 4.3

A function $(\widehat{\mathfrak{u}},\widehat{\vartheta }) \in \mathcal{B}$ is a solution of inequalities (4.1) and (4.2) if and only if there exist functions $z_{1},z_{2}\in \mathcal{PC} (J)$ such that

(i) $$\bigl\{\scriptsize \begin{array}{c} \vert z_{1}(\varkappa) \vert \leq \epsilon _{1},\text{ } \sigma \in J, \\ \vert z_{2}(\varkappa) \vert \leq \epsilon _{2},\text{ } \sigma \in J,\end{array}$$

(ii) $$\bigl\{\scriptsize \begin{array}{c} \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}}(\sigma)=f(\sigma ,\widehat{\mathfrak{u}}(\sigma), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(\sigma))+z_{1}(\sigma),\text{ }\sigma \in J, \\ \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(\sigma)=g(\sigma ,\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\widehat{\mathfrak{u}}(\sigma), \widehat{\vartheta }(\sigma))+z_{2}(\sigma),\text{ }\sigma \in J.\end{array}$$

Lemma 4.4

Let $\mathfrak{y}\in (0,1)$ *and* $\mathfrak{p}\in [0,1]$. *If a function* $(\widehat{\mathfrak{u}},\widehat{\vartheta }) \in \mathcal{B}$ *satisfies inequalities* (4.1) *and* (4.2), *then* $(\widehat{\mathfrak{u}},\widehat{\vartheta })$ *satisfies the following integral inequalities*:

$$\textstyle\begin{cases} \vert \widehat{\mathfrak{u}}(\sigma)-\mathcal{A}_{ \widehat{\mathfrak{u}}}-\mathcal{I}_{\sigma$$

\widehat{\mathfrak{u}}(s),\mathcal{D}_{ [\sigma
] }^{ \mathfrak{y,p,\varphi }}\widehat{\vartheta }
(s)) (\sigma) \vert \leq \epsilon _{1}K, \\ \vert
\widehat{\vartheta }(\sigma)-\mathcal{A}_{
\widehat{\vartheta }}-\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi
}g(s,\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}
\widehat{\mathfrak{u}}(s),\widehat{\vartheta }
(s),) (\sigma) \vert \leq \epsilon
_{2}K,\end{cases}

where
\textstyle\begin{cases}
\mathcal{A}_{\widehat{\mathfrak{u}}}:=\sum_{0<
\sigma _{k}< \sigma } \frac{ (\varphi (\sigma
_{k})-\varphi (\sigma _{k-1}))
^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi
(\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} [
w_{1}-\mathcal{I}_{\sigma _{k-1}^{+}}^{
\mathfrak{y},\varphi }f(s,\widehat{\mathfrak{u}}
(s),\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\widehat{\vartheta }
(s)) (T)] \\
\hphantom{\mathcal{A}_{\widehat{\mathfrak{u}}}:=}
{}-\sum_{0< \sigma _{k}< \sigma
}\mathcal{I}_{\sigma _{k-1}^{+}}^{
\mathfrak{y},\varphi }f(s,\widehat{\mathfrak{u}}
(s),\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\widehat{\vartheta }
(s)) (\sigma _{k}) -\sum_{0< \sigma _{k}<
\sigma }Z_{k} \widehat{\mathfrak{u}}(\sigma
_{k}^{-}), \\ \mathcal{A}_{\widehat{\vartheta
}}:=\sum_{0< \sigma _{k}< \sigma } \frac{ (
[illegible]}))
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^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi

(\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} [
w_{2}-\mathcal{I}_{\sigma _{k-
1}^{+}}^{\mathfrak{y},\varphi
}g(s,\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}
\widehat{\mathfrak{u}}(s),\widehat{\vartheta }
(s)) (T)] \\
\hphantom{\mathcal{A}_{\widehat{\vartheta
}}:=}{}-\sum_{0< \sigma _{k}< \sigma
}\mathcal{I}_{\sigma _{k-1}^{+}}^{
\mathfrak{y},\varphi }g(s,\mathcal{D}_{ [\sigma
] }^{ \mathfrak{y,p,\varphi
}}\widehat{\mathfrak{u}}(s),\widehat{\vartheta }
(s)) (\sigma _{k}) -\sum_{0< \sigma _{k}<
\sigma }Z_{k}\widehat{\vartheta }(\sigma
_{k}^{-}),\end{cases}

and

K:= (2m+1) \frac{ (\varphi (T)-\varphi (0))
^{\mathfrak{y}}}{\Gamma (\mathfrak{y}+1)}.

Proof

Indeed, by Remark 4.3 we have

\textstyle\begin{cases} \mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}
\widehat{\mathfrak{u}}(\sigma)=f(\sigma
,\widehat{\mathfrak{u}}(\sigma), \mathcal{D}_{
[\sigma] }^{\mathfrak{y,p,\varphi }}
\widehat{\vartheta }(\sigma))+z_{1}(\sigma
),&\sigma \in J, \\ \mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }
(\sigma)=g(\sigma ,\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi
}}\widehat{\mathfrak{u}}(\sigma),
\widehat{\vartheta }(\sigma))+z_{2}(\sigma
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),&\sigma \in J.\end{cases}

Then, for \sigma \in (\sigma _{k},\sigma _{k+1}] , k=1,\ldots,m, we get \textstyle\begin{cases} \vert \widehat{\mathfrak{u}}(\sigma)-\mathcal{A}_{ \widehat{\mathfrak{u}}}-\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }f(s, \widehat{\mathfrak{u}} (s),\mathcal{D}_{0}^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(s)) (\sigma) \vert \\ \quad \leq \sum_{0< \sigma _{k}< \sigma } \frac{ (\varphi (\sigma _{k})-\varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} [\mathcal{I}_{\sigma _{k-1}^{+}}^{\mathfrak{y},\varphi } \vert z_{1}(s) \vert (T)] \\ \qquad {}+\sum_{0< \sigma _{k}< \sigma }\mathcal{I}_{\sigma _{k-1}^{+}}^{ \mathfrak{y},\varphi } \vert z_{1}(s) \vert (\sigma _{k}) +\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi } \vert z_{1}(s) \vert (\sigma),\\ \vert \widehat{\vartheta } (\sigma)-\mathcal{A}_{ \widehat{\vartheta }}-\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }g(s,\mathcal{D}_{0}^{\mathfrak{y,p,\varphi }}\widehat{\mathfrak{u}}(s), \widehat{\vartheta } (s)) (\sigma) \vert \\ \quad \leq \sum_{0< \sigma _{k}< \sigma } \frac{ (\varphi (\sigma _{k})- \varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}}{ (\varphi (T)-\varphi (\sigma _{k-1})) ^{\mathfrak{\gamma }-1}} [\mathcal{I}_{\sigma _{k-1}^{+}}^{\mathfrak{y},\varphi } \vert z_{2} (s) \vert (T)] \\ \qquad {}+\sum_{0< \sigma _{k}< \sigma }\mathcal{I}_{\sigma _{k-1}^{+}}^{

Loading [MathJax]/jax/output/SVG/autoload/mtable.js \sigma _{k}) +\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi } \vert z_{2}(s) \vert (\sigma).\end{cases}

It follows that

\textstyle\begin{cases} \vert \widehat{\mathfrak{u}}(\sigma)-\mathcal{A}_{ \widehat{\mathfrak{u}}}-\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }f(s, \widehat{\mathfrak{u}}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{\vartheta } (s)) (\sigma) \vert \leq \epsilon _{1}K, \\ \vert \widehat{\vartheta }(\sigma)-\mathcal{A}_{ \widehat{\vartheta }}-\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }g(s,\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}}(s),\widehat{\vartheta } (s),) (\sigma) \vert \leq \epsilon _{2}K.\end{cases}

□

In the forthcoming theorem, we prove the stability results for system (1.1).

Theorem 4.5

Assume that (H_{1}) *and* (H_{2}) *hold. Then* \textstyle\begin{cases} \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \mathfrak{u}(\sigma)=f(\sigma ,\mathfrak{u}(\sigma),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta (\sigma)), &\sigma \in J, \\ \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \vartheta (\sigma)=g(\sigma ,\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma),\vartheta (\sigma)),& \sigma \in J, \end{cases}

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(4.5)

are UH stable, provided that $\Delta = (1-\Lambda _{1f}) (1-\Lambda _{2g}) -\Lambda _{1g}\Lambda _{2f}\neq 0$, *where*

$$\begin{aligned} \Lambda _{1f} =&\frac{\varrho _{f} (\varphi (T)-\varphi (0)) ^{\mathfrak{y}}}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) }, \qquad \Lambda _{2f}= \frac{ (\varphi (T)-\varphi (0)) ^{\mathfrak{y}}\varrho _{f}\varrho _{g}}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) }, \\ \Lambda _{1g} =&\frac{ (\varphi (T)-\varphi (0)) ^{\mathfrak{y}}\varrho _{f}^{\prime }\varrho _{g}^{\prime }}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) },\qquad \Lambda _{2g}= \frac{ (\varphi (T)-\varphi (0)) ^{\mathfrak{y}}\varrho _{g}^{\prime }}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) }. \end{aligned}$$

Proof

Let $\epsilon =\max \{\epsilon _{1},\epsilon _{2}\}>0$, let $(\widehat{\mathfrak{u}},\widehat{\vartheta }) \in \mathcal{B}$ be functions satisfying inequalities (4.1) and (4.2), and let $(\mathfrak{u},\vartheta) \in \mathcal{B}$ be the unique solution of the following system

\textstyle\begin{cases} \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \mathfrak{u}(\sigma)=f(\sigma ,\mathfrak{u}(\sigma),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }}\vartheta

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\sigma _{k},k=1,\ldots,m, \\ \mathcal{D}_{ [
\sigma] }^{\mathfrak{y,p,\varphi }} \vartheta
(\sigma)=g(\sigma ,\mathcal{D}_{ [\sigma] }^{
\mathfrak{y,p,\varphi }}\mathfrak{u}(\sigma
),\vartheta (\sigma)),\quad\sigma \in J:= [0,T]
,\sigma \neq \sigma _{k}, k=1,\ldots,m, \\ \Delta
\mathfrak{u} \vert _{\sigma =\sigma _{k}}=
\Delta \widehat{\mathfrak{u}} \vert _{\sigma
=\sigma _{k}}=Z_{k} \widehat{\mathfrak{u}}
(\sigma _{k}^{-}),\quad k=1,\ldots,m, \\ \Delta
\vartheta \vert _{\sigma =\sigma _{k}}= \Delta
\widehat{\vartheta } \vert _{\sigma =\sigma
_{k}}=Z_{k} \widehat{\vartheta }(\sigma
_{k}^{-}),\quad k=1,\ldots,m, \\
\widehat{\mathfrak{u}}(T)=\mathfrak{u}
(T)=w_{1},\qquad \widehat{\vartheta }
(T)=\vartheta (T)=w_{2}. \end{cases}

Then by Theorem 3.1 we have
\textstyle\begin{cases} \mathfrak{u}(\sigma
)=\mathcal{A}_{\mathfrak{u}}+\mathcal{I}_{\sigma
_{k}^{+}}^{ \mathfrak{y},\varphi
}f(s,\mathfrak{u}(s),\mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\vartheta (s)) (\sigma
), \\ \vartheta (\sigma)=\mathcal{A}_{\vartheta
}+\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi
}g(s,\mathcal{D}_{ [\sigma] }^{
\mathfrak{y,p,\varphi }}\mathfrak{u}(s),\vartheta
(s)) (\sigma).\end{cases}

Since
\textstyle\begin{cases} \Delta \mathfrak{u} \vert
_{\sigma =\sigma _{k}}= \Delta
\widehat{\mathfrak{u}} \vert _{\sigma =\sigma
_{k}}=Z_{k} \widehat{\mathfrak{u}}(\sigma

_{k}^{-}),\quad k=1,\ldots,m, \\ \Delta \vartheta \vert _{\sigma =\sigma _{k}}= \Delta \widehat{\vartheta } \vert _{\sigma =\sigma _{k}}=Z_{k} \widehat{\vartheta }(\sigma _{k}^{-}),\quad k=1,\ldots,m, \\ \widehat{\mathfrak{u}}(T)=\mathfrak{u}(T)=w_{1}, \\ \widehat{\vartheta }(T)=\vartheta (T)=w_{2},\end{cases}

we can easily prove that \mathcal{A}_{\mathfrak{u}}=\mathcal{A}_{\widehat{\mathfrak{u}}} and \mathcal{A}_{\vartheta }=\mathcal{A}_{\widehat{\vartheta }}. Hence from (H_2) and Lemma 4.4, for each \sigma \in J, we have

\begin{aligned} \bigl\vert \widehat{\mathfrak{u}}(\sigma)-\mathfrak{u}(\sigma) \bigr\vert =&\bigl\vert \widehat{\mathfrak{u}}(\sigma)-\mathcal{A}_{\widehat{\mathfrak{u}}}-\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }f\bigl(s,\mathfrak{u}(s), \mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\vartheta (s)\bigr) (\sigma) \\ &- \mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }f\bigl(s, \widehat{\mathfrak{u}}(s),\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{\vartheta }(s)\bigr) (\sigma) +f\bigl(s,\widehat{\mathfrak{u}}(s), \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(s) \bigr) (\sigma) \bigr\vert \\ \leq & \bigl\vert \widehat{\mathfrak{u}}(\sigma)-\mathcal{A}_{ \widehat{\mathfrak{u}}}- \mathcal{I}_{\sigma

}f\bigl(s,\widehat{ \mathfrak{u}}

(s),\mathcal{D}_{ [\sigma] }^{
\mathfrak{y,p,\varphi }}\widehat{\vartheta }
(s)\bigr) (\sigma) \bigr\vert \\
&+\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert f
\bigl(s,\widehat{\mathfrak{u}}(s),\mathcal{D}_{ [
\sigma] }^{ \mathfrak{y,p,\varphi }}
\widehat{\vartheta }(s)\bigr) (\sigma) -f\bigl(s,
\mathfrak{u}(s), \mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }} \vartheta (s)\bigr) (
\sigma) \bigr\vert \\ \leq &K\epsilon
_{1}++\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y}, \varphi } \bigl\vert
f\bigl(s,\widehat{\mathfrak{u}}(s),\mathcal{D}_{
[\sigma] }^{\mathfrak{y,p,\varphi }}
\widehat{\vartheta }(s)\bigr) (\sigma) -
f\bigl(s,\mathfrak{u}(s), \mathcal{D}_{ [\sigma]
}^{\mathfrak{y,p,\varphi }}\vartheta (s)\bigr) (
\sigma) \bigr\vert \end{aligned}

(4.6)

and
\begin{aligned} \bigl\vert \widehat{\vartheta }
(\sigma)-\vartheta (\sigma) \bigr\vert =&
\bigl\vert \widehat{\vartheta }(\sigma)-
\mathcal{A}_{ \widehat{\vartheta }}-
\mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi }g\bigl(s,
\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi
}} \mathfrak{u}(s), \vartheta (s)\bigr) (\sigma) \\
&- \mathcal{I}_{\sigma
_{k}^{+}}^{\mathfrak{y},\varphi }g\bigl(s,
\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi
theta }
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(s)\bigr) (\sigma) +g\bigl(s, \mathcal{D}_{ [

\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}}(s), \widehat{\vartheta } (s)\bigr) (\sigma) \bigr\vert \\ \leq & \bigl\vert \widehat{\vartheta }(\sigma)-\mathcal{A}_{ \widehat{\vartheta }}- \mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi }g\bigl(s,\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}}(s),\widehat{\vartheta } (s)\bigr) (\sigma) \bigr\vert \\ &+\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y},\varphi } \bigl\vert g \bigl(s,\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}}(s),\widehat{\vartheta } (s)\bigr) (\sigma) -g \bigl(s, \mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \mathfrak{u} (s),\vartheta (s)\bigr) (\sigma) \bigr\vert \\ \leq &K\epsilon _{2}+\mathcal{I}_{\sigma _{k}^{+}}^{\mathfrak{y}, \varphi } \bigl\vert g\bigl(s,\mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{ \mathfrak{u}} (s),\widehat{\vartheta }(s)\bigr) (\sigma) - g\bigl(s, \mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\mathfrak{u}(s),\vartheta (s)\bigr) (\sigma) \bigr\vert . \end{aligned}

(4.7)

Thus by (H_1) we have

\Vert \widehat{\mathfrak{u}}-\mathfrak{u} \Vert _{ \mathcal{PC} (J) }\leq K \epsilon _{1}+ \biggl[\frac{\varrho _{f} \Vert \widehat{\mathfrak{u}}- \mathfrak{u} \Vert _{\mathcal{PC} (J) }+\varrho

\widehat{\vartheta }-\vartheta \Vert

_{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \biggr] \bigl(\varphi (T)-\varphi (0) \bigr) ^{\mathfrak{y}}.

By the same technique we get

$$\Vert \widehat{\vartheta }-\vartheta \Vert _{ \mathcal{PC} (J) }\leq K\epsilon _{2}+ \biggl[\frac{\varrho _{f}\varrho _{g} \Vert \widehat{\mathfrak{u}}-\mathfrak{u} \Vert _{\mathcal{PC} (J) }+\varrho _{g}^{\prime } \Vert \widehat{\vartheta }-\vartheta \Vert _{\mathcal{PC} (J) }}{\Gamma (\mathfrak{y}+1) (1-\varrho _{f}^{\prime }\varrho _{g}) } \biggr] \bigl(\varphi (T)- \varphi (0) \bigr) ^{\mathfrak{y}}.$$

It follows that

$$\Vert \widehat{\mathfrak{u}}-\mathfrak{u} \Vert _{ \mathcal{PC} (J) } (1- \Lambda _{1f}) - \Vert \widehat{\vartheta }-\vartheta \Vert _{\mathcal{PC} (J) } \Lambda _{1g}\leq K\epsilon _{1} \tag{4.8}$$

and

$$\Vert \widehat{\vartheta }-\vartheta \Vert _{ \mathcal{PC} (J) } (1-\Lambda _{2g}) - \Vert \widehat{\mathfrak{u}}-\mathfrak{u} \Vert _{\mathcal{PC} (J) }\Lambda _{2f} \leq K\epsilon _{2}. \tag{4.9a}$$

Inequalities (4.8) and (4.9a) can be rewritten in the matrix form

$$\begin{pmatrix} (1-\Lambda _{1f}) & -\Lambda _{1g} \\ -\Lambda _{2f} & (1-\Lambda _{2g})\end{pmatrix} \begin{pmatrix} \Vert \widehat{\mathfrak{u}}-\mathfrak{u} \Vert _{ \mathcal{PC} (J) } \\ \Vert \widehat{\vartheta }-\vartheta \Vert _{ \mathcal{PC} (J) }\end{pmatrix} \leq \begin{pmatrix} \epsilon _{1}K \\ \epsilon _{2}K\end{pmatrix} .$$

By simple computations this inequality becomes

$$\begin{pmatrix} \Vert \widehat{\mathfrak{u}}-\mathfrak{u} \Vert _{ \mathcal{PC} (J) } \\ \Vert \widehat{\vartheta }-\vartheta \Vert _{ \mathcal{PC} (J) }\end{pmatrix} \leq \frac{1}{\Delta }\begin{pmatrix} (1-\Lambda _{2g}) & \Lambda _{1g} \\ \Lambda _{2f} & (1-\Lambda _{1f})\end{pmatrix} \times \begin{pmatrix} \epsilon _{1}K \\ \epsilon _{2}K\end{pmatrix} .$$

This leads to

$$\begin{aligned}& \Vert \widehat{\mathfrak{u}}-\mathfrak{u} \Vert _{ \mathcal{PC} (J) } \leq \frac{ (1-\Lambda _{2g}) \epsilon _{1}K+\Lambda _{1g}\epsilon _{2}K}{\Delta }, \\& \Vert \widehat{\vartheta }-\vartheta \Vert _{ \mathcal{PC} (J) } \leq \frac{\Lambda _{2f}\epsilon _{1}K+ (1-\Lambda _{1f}) \epsilon _{2}K}{\Delta }. \end{aligned}$$

Thus

\begin{aligned} \bigl\Vert (\widehat{\mathfrak{u}},\widehat{\vartheta }) - (\mathfrak{u}, \vartheta) \bigr\Vert _{

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\mathfrak{u}}-\mathfrak{u} \Vert _{\mathcal{PC} (J) }+ \Vert \widehat{\vartheta }-\vartheta \Vert _{\mathcal{PC} (J) } \\ \leq &\frac{2-\Lambda _{2g}+\Lambda _{1g}+\Lambda _{2f}-\Lambda _{1f}}{\Delta }\epsilon K \\ \leq &\mathcal{M}\epsilon , \end{aligned}

(4.10)

where \epsilon =\max \{\epsilon _{1},\epsilon _{2}\} and \mathcal{M}= \frac{2-\Lambda _{2g}+\Lambda _{1g}+\Lambda _{2f}-\Lambda _{1f}}{\Delta }K. Hence by inequality (4.10) and Definition 4.1 the solution of system (1.1) is Ulam–Hyers stable. Next, by setting \lambda _{\phi }=\epsilon \mathcal{M} such that \lambda _{\phi }(0)=0 system (1.1) is generalized Ulam–Hyers stable. □

5 An example

Consider the following problem: \textstyle\begin{cases} \mathcal{D}_{ [\sigma] }^{\mathfrak{y},\mathfrak{p}, \varphi }\mathfrak{u}(\sigma)= \frac{ (\varphi (\sigma)-\varphi (\frac{1}{5})) ^{\mathfrak{\gamma -}1}}{10e^{ (\varphi (\sigma)-\varphi (0)) }} [\frac{ \vert \mathfrak{u}(\sigma) \vert }{ (1+ \vert \mathfrak{u}(\sigma) \vert) }+ \frac{ \vert \mathcal{D}_{ [\sigma] }^{\mathfrak{y},\mathfrak{p},\varphi }\vartheta (\sigma) \vert }{ (1+ \vert \mathcal{D}_{ [\sigma] }^{\mathfrak{y},\mathfrak{p},\varphi }\vartheta (\sigma) \vert) }] ,\quad \sigma \in (0,1] - \{ \frac{1}{5} \} , \\ \mathcal{D}_{ [\sigma] }^{\mathfrak{y},\mathfrak{p}, \varphi }\vartheta

(\frac{1}{5})) ^{\mathfrak{\gamma -}1}}{10e^{ (\varphi (\sigma)-\varphi (0)) }} [\frac{ \vert \mathcal{D}_{ [\sigma] }^{\mathfrak{y},\mathfrak{p},\varphi }\mathfrak{u}(\sigma) \vert }{ (1+ \vert \mathcal{D}_{ [\sigma] }^{\mathfrak{y},\mathfrak{p},\varphi }\mathfrak{u}(\sigma) \vert) }+ \frac{ \vert \vartheta (\sigma) \vert }{ (1+ \vert \vartheta (\sigma) \vert) }] ,\quad \sigma \in (0,1] - \{ \frac{1}{5} \} , \\ \Delta \mathfrak{u}(\frac{1}{5}^{-})= \frac{ \vert \mathfrak{u}(\frac{1}{5}^{-}) \vert }{8(1+ \vert \mathfrak{u}(\frac{1}{5}^{-}) \vert)}, \qquad \Delta \vartheta (\frac{1}{5}^{-})= \frac{ \vert \vartheta (\frac{1}{5}^{-}) \vert }{8(1+ \vert \vartheta (\frac{1}{5}^{-}) \vert)}, \\ \mathfrak{u}(1)=3,\qquad \vartheta (1)=2,\end{cases}

(5.1)

Here \mathfrak{y}=\frac{1}{3}, \mathfrak{p}=\frac{1}{2}, \mathfrak{\gamma }=\frac{2}{3}, w_{1}=3, w_{2}=2. Set \varphi (\sigma)=e^{\sigma } .

Example 5.1

Define f,g: (0,1] \times \mathbb{R} ^{2}\rightarrow \mathbb{R} as \begin{aligned}& f\bigl(\sigma ,\mathfrak{u} (\sigma),\mathcal{D}_{0}^{ \mathfrak{y,p,\varphi }} \vartheta (\sigma)\bigr)= \frac{ (\varphi (\sigma)-\varphi (\frac{1}{5})) ^{\mathfrak{\gamma -}1}}{10e^{ (\varphi (\sigma)-\varphi (0)) }} \biggl[\frac{ \vert

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\mathfrak{u}(\sigma) \vert }{ (1+ \vert

\mathfrak{u}(\sigma) \vert) }+ \frac{ \vert
\mathcal{D}_{ [\sigma]
}^{\mathfrak{y},\mathfrak{p},\varphi }\vartheta
(\sigma) \vert }{ (1+ \vert \mathcal{D}_{ [\sigma
] }^{\mathfrak{y},\mathfrak{p},\varphi }\vartheta
(\sigma) \vert) } \biggr] , \\& g\bigl(\sigma
,\mathcal{D}_{0}^{\mathfrak{y,p,\varphi
}}\mathfrak{u}(\sigma),\vartheta (\sigma
)\bigr)= \frac{ (\varphi (\sigma)-\varphi (\frac{1}
{5})) ^{\mathfrak{\gamma -}1}}{10e^{ (\varphi
(\sigma)-\varphi (0)) }} \biggl[\frac{ \vert
\mathcal{D}_{ [\sigma]
}^{\mathfrak{y},\mathfrak{p},\varphi
}\mathfrak{u}(\sigma) \vert }{ (1+ \vert
\mathcal{D}_{ [\sigma]
}^{\mathfrak{y},\mathfrak{p},\varphi
}\mathfrak{u}(\sigma) \vert) }+ \frac{ \vert
\vartheta (\sigma) \vert }{ (1+ \vert \vartheta
(\sigma) \vert) } \biggr] , \end{aligned}

and Z_{1},Z_{2}:\mathbb{R} \rightarrow
\mathbb{R} by
Z_{1}(\mathfrak{u})= \frac{ \vert \mathfrak{u}
\vert }{8(1+ \vert \mathfrak{u} \vert)}

and
Z_{2}(\vartheta)= \frac{ \vert \vartheta \vert }
{8(1+ \vert \vartheta \vert)}.

Then, for (\mathfrak{u},\vartheta) ,
(\widehat{\mathfrak{u}},\widehat{\vartheta })
\in\mathbb{R} \times\mathbb{R} , we have
\begin{aligned} \bigl\vert f\bigl(\sigma
,\mathfrak{u} (\sigma) ,\vartheta (\sigma)
\bigr)-f\bigl(\sigma ,\widehat{\mathfrak{u}} (
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gr)

\bigr\vert \leq &\frac{1}{10} \bigl\vert \mathfrak{u} (\sigma) - \widehat{\mathfrak{u}} (\sigma) \bigr\vert + \frac{1}{10} \bigl\vert \vartheta (\sigma) - \widehat{ \vartheta } (\sigma) \bigr\vert , \\ \bigl\vert g\bigl(\sigma ,\mathfrak{u} (\sigma) ,\vartheta (\sigma) \bigr)-g\bigl(\sigma ,\widehat{\mathfrak{u}} (\sigma) ,\widehat{ \vartheta } (\sigma) \bigr) \bigr\vert \leq &\frac{1}{10} \bigl\vert \mathfrak{u} (\sigma) - \widehat{\mathfrak{u}} (\sigma) \bigr\vert + \frac{1}{10} \bigl\vert \vartheta (\sigma) - \widehat{ \vartheta } (\sigma) \bigr\vert , \end{aligned}

and

\begin{aligned}& \bigl\vert Z_{1}(\mathfrak{u})-Z_{1}(\widehat{ \mathfrak{u}}) \bigr\vert \leq \frac{1}{8} \bigl\vert \mathfrak{u} (\sigma) - \widehat{\mathfrak{u}} (\sigma) \bigr\vert , \\& \bigl\vert Z_{2}(\vartheta)-Z_{2}(\widehat{ \vartheta }) \bigr\vert \leq \frac{1}{8} \bigl\vert \vartheta (\sigma) - \widehat{\vartheta } (\sigma) \bigr\vert . \end{aligned}

Here \varrho _{f}=\varrho _{f}^{\prime }=\varrho _{g}=\varrho _{g}^{ \prime }=\omega _{f}=\omega _{f}^{\prime }=\omega _{g}=\omega _{g}^{ \prime }=\frac{1}{10} and L_{Z_{1}}=L_{Z_{2}}=\frac{1}{8}. From the given data we deduce that conditions (H_{1}), (H_{2}), and (H_{3}) hold. Thus all the conditions of Theorem 3.1 are satisfied. Therefore problem (1.1) has at least one solution on [0,1] . Moreover, we have \rho _{1}=\rho _{2}=0,1 and \mathcal{O}=0.48<1 Thus all conditions of

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Theorem 3.2 are satisfied. Therefore problem (1.1) has a unique solution on $[0,1]$.

Finally, for $\epsilon =\max \{\epsilon _{1},\epsilon _{2}\}>0$, we find that the inequalities

$$\begin{aligned}& \bigl\vert \mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{ \mathfrak{u}}(\sigma)-f\bigl(\sigma ,\widehat{\mathfrak{u}}(\sigma),\mathcal{D}_{ [\sigma] }^{\mathfrak{y,p,\varphi }} \widehat{\vartheta }(\sigma)\bigr) \bigr\vert \leq \epsilon _{1}, \\& \bigl\vert \mathcal{D}_{ [\sigma] }^{ \mathfrak{y,p,\varphi }}\widehat{ \vartheta }(\sigma)-f\bigl(\sigma ,\mathcal{D}_{0}^{ \mathfrak{y,p,\varphi }} \widehat{\mathfrak{u}}(\sigma),\widehat{\vartheta }(\sigma)\bigr) \bigr\vert \leq \epsilon _{2} \end{aligned}$$

are satisfied. Then equation (4.5) is Ulam–Hyers stable with

$$\bigl\Vert (\widehat{\mathfrak{u}},\widehat{\vartheta }) - (\mathfrak{u},\vartheta) \bigr\Vert _{\mathcal{B}}\leq \mathcal{M}\epsilon,\quad \varkappa \in J.$$

where $\mathcal{M}=2.3>0$.

6 Concluding remarks

We obtained the existence, uniqueness, and UH stability of solutions for a new problem of φ-Hilfer FDEs with impulse conditions. Our investigations were based on the reduction of FDEs to FIEs and application the standard Leray–Schauder and

 sults

in this paper are more general and cover many of the parallel problems that contain particular cases of the function φ, because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives; for instance, for various values of a function φ and parameter \mathfrak{p}, the coupled system (1.1) includes coupled systems of FDEs involving the Hilfer, Hadamard, Katugampola, and many other fractional derivative operators, which are described in the introduction.

Availability of data and materials

The authors declare that all data and materials in this paper are available and veritable.

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Author information

Authors and Affiliations

Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, (M.S), 431001, India
Mohammed A. Almalahi & Satish K. Panchal

Department of Mathematics, Hajjah University, Hajjah, Yemen
Mohammed A. Almalahi

Contributions

MAA: Writing original draft, conceptualization, writing review and editing, methodology. SKP: Supervision, review, and editing. All authors read and approved the final manuscript.

Corresponding author

Correspondence to Mohammed A. Almalahi.

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