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Some properties of implicit impulsive coupled system via φ -Hilfer fractional operator

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Abstract

The major goal of this work is investigating sufficient conditions for the existence and uniqueness of solutions for implicit impulsive coupled system of φ -Hilfer fractional differential equations (FDEs) with instantaneous impulses and terminal conditions. First, we derive equivalent fractional integral equations of the proposed system. Next, by employing some standard fixed point theorems such as Leray–Schauder alternative

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uniqueness of solutions. Further, by mathematical analysis technique we investigate the Ulam–Hyers (UH) and generalized UH (GUH) stability of solutions. Finally, we provide a pertinent example to corroborate the results obtained.

1 Introduction

Fractional differential equations (FDEs) have attracted the interest of researchers from various disciplines as they are a useful tool in modeling the dynamics of numerous physical systems and have applications in many fields of applied sciences, engineering and technical sciences, and so on. For further details, see [26, 36, 38, 40]. There are various definitions of fractional calculus (FC) used in FDEs for modeling and describing the memory accurately. Among the famous operators of this calculus, there are Riemann–Liouville, Riemann, Grünwald–Letnikov, Caputo, Hilfer, and Hadamard, which are the most used. For more detail, we refer the readers to [1–3, 21, 22, 24, 25, 33, 34, 36, 41]. There is a prominent and noticeable interest in the investigation of qualitative characteristics of solutions (existence, uniqueness, stability) of FDEs. For applications and recent work, we refer the readers to [4, 7, 14, 18, 37, 42, 43].

In recent years, the impulsive fractional differential equations have become an important and successful tool in modeling some physical phenomena that have sudden changes and have discontinuous jumps by imposing impulsive conditions on the fractional differential equations at discontinuity points. For applications and recent work, we refer

the readers to [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 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On the other side, the study of coupled systems involving FDEs is also important as such systems occur in various problems of applied nature. For some theoretical works on coupled systems of FDEs, we refer to series of papers [11, 16, 19, 20, 23, 30].

The topic of system stability is one of the most important qualitative characteristics of a solution, but to our knowledge, the results on UH and UHR stability of solutions for implicit impulsive coupled systems are very few in the literature.

Very recently, Kharade and Kucche [35] studied the existence and uniqueness of solutions and UHML stability for the following impulsive implicit problem:

```
\textstyle\begin{cases}
\mathcal{D}^{+}\{\mathfrak{y}, \mathfrak{p}; \varphi\} u(\sigma) = f(\sigma, u(\sigma), u(h(\sigma))), \\
\mathcal{D}^{+}\{\mathfrak{y}, \mathfrak{p}; \varphi\} y(\sigma), \sigma \in \mathcal{J} := [0, T], \\
\sigma \neq \sigma_k, k=1, \dots, p, \Delta I_{0^{+}}^{1-\gamma} \varphi u(\sigma_k) \\
= J_k u(\sigma_{k-}), \quad k=1, \dots, p, \Delta I_{0^{+}}^{1-\gamma} \varphi u(0) = u_0, \\
u(\sigma) = \phi(\sigma), \quad \sigma \in [-r, 0],
\end{cases}
```

where

$\mathcal{D}^{+}\{\mathfrak{y}, \mathfrak{p}; \varphi\}$ denotes the φ -Hilfer fractional derivative (FD) of order \mathfrak{y} in

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f: \mathcal{R}^J \times \mathbb{R} \times

$\mathbb{R} \times \mathbb{R} \rightarrow$
 \mathbb{R} is a continuous function. Via standard fixed point theorems, Ahmed et al. [10] studied the existence, uniqueness, and different kinds of stability of the following switched coupled implicit φ -Hilfer fractional differential system:

```
\textstyle\begin{cases}
\mathcal{D}_{a^+}^{\varphi}\mathfrak{y}_p(\sigma) = f(\sigma, \mathfrak{u}(\sigma)) \\
\mathcal{D}_{a^+}^{\varphi}\mathfrak{u}(\sigma) = g(\sigma, \mathfrak{u}(\sigma), \vartheta(\sigma)), \sigma \in J := [0, T], \\
\mathcal{D}_{a^+}^{\varphi}\mathfrak{u}(\sigma) = I_{a^+}^{1-\gamma} \mathfrak{u}(\sigma), \sigma \in J := [0, T], \\
I_{a^+}^{1-\gamma} \mathfrak{u}(a) = \mathfrak{u}(a), \mathfrak{u}(a) \in \mathbb{R}, \vartheta(a) = \vartheta(a) \in \mathbb{R}
\end{cases}
```

where

$\mathcal{D}_{a^+}^{\varphi}\mathfrak{y}_p(\sigma)$ denotes the φ -Hilfer FD of order $\mathfrak{y} \in (0, 1)$ and type $\mathfrak{p} \in [0, 1]$, and $f, g: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Abdo et al. [5], via standard fixed point theorems, studied the existence and uniqueness of the following impulsive problem:

```
\textstyle\begin{cases}
\mathcal{D}_{a^+}^{\varphi}\mathfrak{y}_p(\sigma) = f(\sigma, \mathfrak{u}(\sigma)), \sigma \in J := [0, T], \\
\mathfrak{u}(a) = \mathfrak{u}(a), \vartheta(a) = \vartheta(a)
\end{cases}
```

$$\begin{aligned} \sigma \neq \sigma_{\{k\}, k=1, \dots, m}, \Delta u \\ \text{vert}_{\{\sigma\}} = \sigma_{\{k\}} = I_{\{k\}} u(\sigma_{\{k\}^{\{-\}}}, k=1, \dots, m), \\ u(0)=u_0. \end{aligned}$$

On the other hand, Almalahi et al. [15] studied the existence and uniqueness of solution for the following FDEs:

$$\begin{aligned} \text{textstyle} \begin{cases} \mathcal{D}_{\{a^+\}}^{\{y, p; \varphi\}} y(\sigma) = f(\sigma, y(\sigma)), \\ \mathcal{D}_{\{a^+\}}^{\{y, p; \varphi\}} y(\sigma) , & \sigma \in (a, T], a > 0, \\ y(T) = w \end{cases} \end{aligned}$$

where

$$\mathcal{D}_{\{a^+\}}^{\{y, p; \varphi\}} \text{ is the } \varphi\text{-Hilfer FD of order } y \text{ in } (0, 1) \text{ and type } p \text{ in } [0, 1].$$

Abdo et al. [6] studied the existence, uniqueness, and UH stability of the following system:

$$\begin{aligned} \text{textstyle} \begin{cases} \mathcal{D}_{\{a^+\}}^{\{y_1, p_1; \varphi\}} y_1(\sigma) = f_1(\sigma, y_1(\sigma)), \\ \mathcal{D}_{\{a^+\}}^{\{y_2, p_2; \varphi\}} y_2(\sigma) = f_2(\sigma, y_2(\sigma)) \end{cases} \\ y(T) = w_1, x(T) = w_2 \end{aligned}$$

where

$$\mathcal{D}_{\{a^+\}}^{\{y_1, p_1; \varphi\}}, \mathcal{D}_{\{a^+\}}^{\{y_2, p_2; \varphi\}},$$

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$\text{ak}\{p\}; \mathfrak{\varphi}$ are the φ -Hilfer FDs of orders $\mathfrak{y}_1, \mathfrak{y}_2 \in (0,1)$ and type $\mathfrak{p} \in [0,1]$.

Motivated by the preceding works, in this paper, we investigate the existence, uniqueness, and UH stability for more general implicit impulsive coupled systems of φ -Hilfer FDEs:

$$\begin{aligned} & \text{\textstyle\begin{cases} \mathcal{D}_{\sigma}^{\sigma} [\mathfrak{y}, p, \varphi] \mathfrak{u}(\sigma) = f(\sigma, \mathfrak{u}(\sigma), \mathcal{D}_{\sigma}^{\sigma} [\mathfrak{y}, p, \varphi] \vartheta(\sigma)), \sigma \in J := [0, T], \sigma \neq \sigma_k, k=1, \dots, m, \\ \mathcal{D}_{\sigma}^{\sigma} [\mathfrak{y}, p, \varphi] \vartheta(\sigma) = g(\sigma, \mathcal{D}_{\sigma}^{\sigma} [\mathfrak{y}, p, \varphi] \mathfrak{u}(\sigma), \mathcal{D}_{\sigma}^{\sigma} [\mathfrak{y}, p, \varphi] \vartheta(\sigma)), \sigma \in J := [0, T], \sigma \neq \sigma_k, k=1, \dots, m, \\ \Delta_{\sigma}^{\sigma} \mathfrak{u}(\sigma) = Z_k, \mathfrak{u}(\sigma_k^+) = Z_k, \quad k=1, \dots, m, \\ \mathfrak{u}(T) = w_1, \vartheta(T) = w_2, \end{cases}} \\ & \text{\end{cases}} \end{aligned}$$

(1.1)

where $\mathcal{D}_{\sigma}^{\sigma} [\mathfrak{y}, p, \varphi]$ denotes the φ -Hilfer FD of order \mathfrak{y} in $(0,1)$ and type $\mathfrak{p} \in [0,1]$, $[\sigma] = \sigma_k$ for $\sigma \in (\sigma_k, \sigma_{k+1}], k=0, 1, \dots, m$, $\sigma_0 = 0$. The functions

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 $\rightarrow \mathbb{R}$

$Z_{\{k\}}: \mathbb{R} \rightarrow \mathbb{R}$,
 $k=1,2,\dots,m$, are continuous functions fulfilling
some conditions that will be described later.

Further, $w_{\{1\}}, w_{\{2\}} \in \mathcal{M}(\mathbb{R})$,
 $\sigma_{\{k\}}$ satisfy $0 = \sigma_0 < \sigma_1 < \dots < \sigma_k < \sigma_{k+1} = \sigma$.
 $\Delta \mathfrak{u}'(\sigma) = \mathfrak{u}(\sigma_k^+) - \mathfrak{u}(\sigma_k^-) = \mathfrak{u}(\sigma_k^+) - \mathfrak{u}(\sigma_k)$,
 $\mathfrak{u}(\sigma_k^+) = \lim_{h \rightarrow 0^+} \mathfrak{u}(\sigma_k + h)$,
 $\mathfrak{u}(\sigma_k^-) = \lim_{h \rightarrow 0^-} \mathfrak{u}(\sigma_k + h)$ represent the right and left limits
of $\mathfrak{u}'(\sigma)$ at $\sigma \in (\sigma_k, \sigma_{k+1}]$, $k=0,1,\dots,m$. Δ
 $\vartheta'(\sigma) = \vartheta(\sigma_k^+) - \vartheta(\sigma_k^-) = \vartheta(\sigma_k^+) - \vartheta(\sigma_k)$,
 $\vartheta(\sigma_k^+) = \lim_{h \rightarrow 0^+} \vartheta(\sigma_k + h)$ and $\vartheta(\sigma_k^-) = \lim_{h \rightarrow 0^-} \vartheta(\sigma_k + h)$ represent the right and left limits of
 $\vartheta'(\sigma)$ at $\sigma \in (\sigma_k, \sigma_{k+1}]$, $k=0,1,\dots,m$.

The coupled systems of φ -Hilfer FDEs with
impulsive conditions considered in this work are
a wider class of coupled systems of BVPs that
incorporates the BVPs for FDEs involving the most
broadly used Riemann–Liouville and Caputo
fractional derivatives. Regardless of this, the

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function φ and parameter \mathfrak{p} include coupled systems of FDEs involving the Hilfer, Hadamard, Katugampola, and many other fractional derivative operators.

- If $\varphi(\sigma) = \sigma$ and $\mathfrak{p}=1$, then system (1.1) reduces to an implicit impulsive coupled system with the Caputo fractional derivative.
 - If $\varphi(\sigma) = \sigma$ and $\mathfrak{p}=0$, then system (1.1) reduces to an implicit impulsive coupled system with the Riemann–Liouville fractional derivative.
 - If $\mathfrak{p}=0$, then system (1.1) reduces to an implicit impulsive coupled system with the φ -Riemann–Liouville fractional derivative.
 - If $\varphi(\sigma) = \sigma$, then system (1.1) reduces to an implicit impulsive coupled system with the Hilfer fractional derivative.
 - If $\varphi(\sigma) = \log \sigma$, then system (1.1) reduces to an implicit impulsive coupled system with the Hilfer–Hadamard fractional derivative.
 - If $\varphi(\sigma) = \sigma^{\rho}$, then system (1.1) reduces to an implicit impulsive coupled system with the Katugampola fractional derivative.
- The major contribution of this paper is obtaining an equivalent fractional integral equation of the proposed system and establishing the existence, uniqueness, and UH and GUH stability of a solution for an implicit impulsive coupled system with φ -Hilfer FD. Our analysis relies on the Banach and Leray–Schauder fixed point theorems. Though we use the standard methodology to obtain our results, its exposition to the proposed system is new. The

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general and cover many parallel problems that contain particular cases of functions because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives. Moreover, the results obtained in this work can be extended to n -tuple fractional systems (FSs). Our results include the results of Almalahi et al. [15], Abdo et al. [6], and Kharade et al. [35] and will be a useful contribution to the existing literature on this topic.

This paper is organized as follows. In Sect. 2, we render the rudimentary definitions and prove some lemmas and present some concepts of fixed point theorems. In Sect. 3, we prove the existence and uniqueness of solutions for impulsive implicit coupled system (1.1). In Sect. 4, we discuss the stability by means of mathematical analysis techniques. In Sect. 5, we give a pertinent example illustrating our results. Concluding remarks are presented in the last section.

2 Background material and auxiliary results

In this part, we give important definitions and auxiliary lemmas pertinent to our main results.

Let $J := [0, T]$ and $J^{\prime} := (0, T]$. Let $\mathcal{R} = \mathcal{C}(J)$ be the Banach space of continuous functions $u: J^{\prime} \rightarrow \mathbb{R}$ with the norm $\|u\| = \max_{\sigma \in J} |u(\sigma)|$. Clearly, \mathcal{R} is a Banach space with this norm, and hence the product space $\mathcal{R} \times \mathcal{R}$ is also a Banach space with the norm

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$$\begin{aligned} \|\mathfrak{u}, \vartheta\| &= \\ \|\mathfrak{u}\| + \|\vartheta\| &. \end{aligned}$$

We define the space $\mathcal{PC}(J)$ of piecewise continuous functions $\mathfrak{u}: J^{\prime} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{PC}(J) = & \left\{ \begin{array}{l} \text{continuous } \mathfrak{u} : J^{\prime} \rightarrow \mathbb{R} \\ \mathfrak{u}(\sigma_k), \mathfrak{u}(\sigma_{k+1}) \in \mathbb{R}, k=0,1,\dots,m, \text{ and } \mathfrak{u}(\sigma_k^+) = \mathfrak{u}(\sigma_k^-) \text{ exist for } k=0,1,\dots,m \end{array} \right\} \\ & \text{with } \mathfrak{u}(\sigma_k^+) = \mathfrak{u}(\sigma_k^-) \text{ for } k=0,1,\dots,m. \end{aligned}$$

Obviously, $\mathcal{PC}(J)$ is a Banach space endowed with the norm

$$\|\mathfrak{u}\|_{\mathcal{PC}(J)} = \max_{\sigma \in J} |\mathfrak{u}(\sigma)|.$$

Define the product space

$$\begin{aligned} \mathcal{B} &= \mathcal{PC}(J) \times \mathcal{PC}(J) \\ &\text{with the norm} \\ \|\mathfrak{u}, \vartheta\| &= \|\mathfrak{u}\|_{\mathcal{PC}(J)} + \|\vartheta\|_{\mathcal{PC}(J)} \end{aligned}$$

for $(\mathfrak{u}, \vartheta) \in \mathcal{B}$.

Definition 2.1

([36])

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Let $y > 0$ and $f \in L_{\{1\}}(J)$. Then the generalized RL fractional integral of a function f of order φ with respect to φ is defined as

$$\mathcal{I}_{\{0^+\}}^{\varphi} y(\sigma) = \frac{1}{\Gamma(\varphi)} \int_0^{\sigma} \frac{y(\sigma')}{\varphi'(\sigma') - \varphi(\sigma')} f(\sigma') d\sigma.$$

Definition 2.2

([41])

Let $n-1 < p < n$, and let $f, \varphi \in \mathcal{PC}^n(J)$. Then the generalized Hilfer fractional derivative of a function f of order p and type $0 \leq p \leq 1$ with respect to φ is defined as

$$\begin{aligned} {}^H\mathcal{D}_{\{0^+\}}^p f(\sigma) &= \& \mathcal{I}_{\{0^+\}}^{\varphi} (f(n-\varphi); \varphi) f_{-}(\varphi) \\ &= \& \mathcal{I}_{\{0^+\}}^{\varphi} (f([n]) \mathcal{I}_{\{0^+\}}^{\varphi} ((1-p)(n-\varphi); \varphi) f(\sigma) \\ &= \& \mathcal{I}_{\{0^+\}}^{\varphi} (f([n]) \mathcal{I}_{\{0^+\}}^{\varphi} (n-\varphi; \varphi) f(\sigma) \\ &= \& \mathcal{I}_{\{0^+\}}^{\varphi} (f([n]) \mathcal{I}_{\{0^+\}}^{\varphi} (n-\varphi; \varphi) f(\sigma) \\ &= \& \mathcal{I}_{\{0^+\}}^{\varphi} (f([n]) \mathcal{D}_{\{0^+\}}^p (f(n-\varphi); \varphi)) \\ &= \& \mathcal{D}_{\{0^+\}}^p (f(n-\varphi); \varphi) \\ &= \& \mathcal{D}_{\{0^+\}}^p (f(\sigma); \varphi) \end{aligned}$$

$$\begin{aligned} & \text{\textbackslash mathcal\{D\}_o^{+}}^{\gamma} \text{\textbackslash mathfrak\{\gamma}} \\ & \text{\textbackslash varphi } f(\sigma) = f_{\varphi}^{\gamma}(\sigma)^{[n]} \\ & \text{\textbackslash mathcal\{I\}_o^{+}}^{\gamma}((1-\text{\textbackslash mathfrak\{p\}})(n-} \\ & \text{\textbackslash mathfrak\{y\}); \varphi } f(\sigma), \quad \\ & \text{and} \quad f_{\varphi}^{\gamma}(\sigma)^{[n]} = \biggl(\\ & \frac{1}{\text{\textbackslash prime }(\sigma)} \frac{d}{d\sigma} \biggr)^n. \end{aligned}$$

Lemma 2.3

([41])

Let $\mathfrak{y}' = \mathfrak{y} + \mathfrak{p} - \mathfrak{y}_p$,
 $\mathfrak{y} > 0$, $\mathfrak{p} > 0$, and $u \in$
 $\text{PC}_{[1-\mathfrak{y}]; \varphi}^{\gamma}(\mathbb{J})$. Then
 $\mathcal{I}_0^{\gamma} \mathfrak{y}' =$
 $\mathcal{D}_0^{\gamma} \mathfrak{y} +$
 $\mathcal{I}_0^{\gamma} \mathfrak{y}; \varphi$
 and
 $\mathcal{D}_0^{\gamma} \mathfrak{y}' =$
 $\mathcal{D}_0^{\gamma} \mathfrak{y} +$
 $\mathcal{D}_0^{\gamma} \mathfrak{p} (1 -$
 $\mathfrak{y}); \varphi u.$

Theorem 2.4

([41])

Let $0 \leq \mathfrak{y} < \mathfrak{y}$ and
 $u \in \text{PC}(\mathbb{J})$. Then

$$\mathcal{I}_0^{\gamma} \mathfrak{y} =$$

$\lim_{n \rightarrow \infty}$

 $\mathcal{I}_0^{\gamma} \mathfrak{y} = 0.$

Lemma 2.5

([36, 41])

Let $\mathfrak{y}, \mathfrak{p} > 0$ and $\delta > 0$.

Then

$$\begin{aligned} &\begin{aligned} &\text{\begin{aligned}} \\ &\mathcal{I}_0^+ (\mathfrak{y}, \varphi) \\ &\mathcal{I}_0^+ (\mathfrak{p}, \varphi) f(\\ &\sigma \\ &)=\mathcal{I}_0^+ (\mathfrak{y}+\mathfrak{p}, \\ &\varphi) f(\sigma), \quad \& \\ &\mathcal{I}_0^+ (\mathfrak{y}, \varphi) \\ &\bigl(\varphi(\sigma)-\varphi(0) \bigr)^{\delta} \\ &-1=\frac{\Gamma(\gamma)}{\Gamma(\mathfrak{y}+\gamma)} \\ &\{\Gamma(\mathfrak{y}+\gamma)\} \\ &\bigl(\varphi(\sigma)-\varphi(0) \bigr) \\ &^{\delta-1}, \end{aligned}} \end{aligned}$$

and

$$\begin{aligned} &\begin{aligned} &^H \mathcal{D}_0^+ (\mathfrak{y}, \mathfrak{p}, \varphi) \\ &\bigl(\varphi(\sigma)-\varphi(0) \bigr) \\ &^{\gamma-1}=0, \quad \text{quad} \\ &\mathfrak{y} \\ &=\mathfrak{y}+n\mathfrak{p}-\mathfrak{y}''. \end{aligned} \end{aligned}$$

Lemma 2.6

([41])

If $\mathcal{P}_C^n(J)$, $n-1 < \mathfrak{y} < n$, and $0 \leq \mathfrak{p} \leq 1$, then

$$\begin{aligned} &\mathcal{I}_0^+ (\mathfrak{y}; \varphi) \\ &\mathcal{D}_0^+ (\mathfrak{y}, \mathfrak{p}, \varphi) \\ &f(\sigma)=f(\sigma)-\sum_{k=1}^n \frac{(\varphi(\sigma)-\varphi(0))^{\gamma-k}}{\Gamma(\gamma-k)} \\ &\frac{(\varphi(\sigma)-\varphi(0))^{\gamma-n}}{\Gamma(\gamma-n)} \end{aligned}$$

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$\mathcal{I}_a^{\{+}\}^{(1-\mathfrak{p})(n-\mathfrak{y})};\varphi f(o),$

and

$\{H\}\mathcal{D}_o^{\{+}\}^{\{y,\mathfrak{p},\varphi\}}\mathcal{I}_o^{\{+}\}^{\{y};\varphi f(\sigma)=f(\sigma).$

Lemma 2.7

([31] (Leray–Schauder alternative))

Let $\Xi : \mathcal{X} \rightarrow \mathcal{X}$ be a completely continuous operator, and let $\text{digamma}(\Xi) = \{ y \in \mathcal{X} : y = \Xi(y), y \in [0,1] \}.$ Then either the set $\text{digamma}(\Xi)$ is unbounded, or Ξ has at least one fixed point.

Theorem 2.8

([29] (Banach fixed point theorem))

Let \mathcal{X} be a Banach space, let $K \subsetneq \mathcal{X}$ be closed, and let $\Xi : K \rightarrow K$ be a strict contraction, that is, $\| \Xi(x) - \Xi(y) \| \leq L \| x - y \|$ for some $0 < L < 1$ and all $x, y \in K.$ Then Ξ has a fixed point in $K.$

Lemma 2.9

Let $\gamma = \frac{y}{p} + \frac{p}{y} - \frac{1}{p},$ $y \in (0,1), p \in [0,1],$ and let $\varphi : J^{\prime} \rightarrow \mathbb{R}$ be a continuous function. Then $u \in \mathcal{PC}^{\{\gamma\}}(J)$ satisfies $\text{D}_{\sigma}^{\{y,p,\varphi\}} u(\sigma) = \varphi(\sigma), \& \sigma \in J := [0,T], \sigma \nearrow \sigma_k, k=1, \dots, m, \Delta Z_k$

$$\begin{aligned} & \mathfrak{u}(\sigma_k^{-})_{k=1}^{\infty}, m, \\ & \mathfrak{u}(T)=w \end{aligned}$$

(2.1)

if and only if \mathfrak{u} satisfies the following integral equations:

$$\begin{aligned} & \mathfrak{u}(\sigma) = \text{cases} \\ & \frac{(\varphi(\sigma) - \varphi(0))}{\Gamma(\gamma_1)} \{ (\varphi(T) - \varphi(0))^{\gamma_1} [w - \\ & \mathcal{I}_0^+ \{ \varphi(s)(T) \}] \\ & + \mathcal{I}_0^+ \{ \varphi(s)(\sigma), \quad \sigma \in [0, \sigma_1], \\ & \sum_{i=1}^{k+1} \frac{(\varphi(\sigma_i) - \varphi(\sigma_{i-1}))}{\Gamma(\gamma_i)} \\ & ^{\gamma_1} \{ (\varphi(\sigma_{i-1}) - \varphi(\sigma_{i-1})) \} \\ & + \sum_{i=1}^k \mathcal{I}_{\sigma_i}^+ \{ \varphi(s)(\sigma_i) \} \\ & + \sum_{i=1}^k Z_{\sigma_i} \mathfrak{u}(\sigma_i^{\gamma_i}), \quad \sigma \in (\sigma_k, \sigma_{k+1}], k=1, \dots, m. \end{aligned}$$

(2.2)

Proof

First, let $\mathfrak{u} \in$
 $\mathcal{PC}^{\gamma}(\mathbb{R})$ be a

solution of problem (2.1). We prove that

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 \mathfrak{u} is a solution of (2.2).

If $\sigma \in [0, \sigma_1]$, then $\mathcal{D}_\sigma^{\sigma} [\varphi(\sigma)] = \varphi(\sigma)$, $\sigma = 0$. Taking the operator

$\mathcal{I}_0^{\sigma} \{y'\}$ on both sides of the first equation in (2.1) and using Lemma 2.6, we have

$$\begin{aligned} \mathfrak{u}(\sigma) &= \frac{(\varphi(\sigma) - \varphi(0))}{\Gamma(\gamma)} + \mathcal{I}_0^{\sigma} \{y'\} \\ &= (\varphi(\sigma) - \varphi(0)) + \mathcal{I}_0^{\sigma} \{y'\} \varphi(s)(\sigma). \end{aligned}$$

(2.3)

By the terminal condition we have

$$\begin{aligned} \mathcal{I}_0^{\sigma} \{y'\} \varphi(s)(T) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma)} (\varphi(T) - \varphi(0)) \\ &\quad + \mathcal{I}_0^{\sigma} \{y'\} \varphi(s)(T). \end{aligned}$$

(2.4)

Putting (2.4) into (2.3), we get

$$\begin{aligned} \mathfrak{u}(\sigma) &= \frac{(\varphi(\sigma) - \varphi(0))}{\Gamma(\gamma)} + \frac{(\varphi(T) - \varphi(0))}{\Gamma(\gamma)} \\ &\quad + \mathcal{I}_0^{\sigma} \{y'\} \varphi(s)(T) + \mathcal{I}_0^{\sigma} \{y'\} \varphi(s)(\sigma). \end{aligned}$$

This means

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$$\begin{aligned} \mathfrak{u}'(\sigma_1^{-}) &= \frac{(\varphi(\sigma_1) - \varphi(0))}{\Gamma(\gamma_1)} \cdot \\ &\quad {}^{\varphi}\text{H}_{\sigma_1}^{\gamma_1} (\varphi(s)(T) - \varphi(s))^{1-\gamma_1} \cdot \\ &\quad {}^{\varphi}\text{I}_{\sigma_1}^{\alpha_1} [\varphi(s)(T) - \varphi(s)] + {}^{\varphi}\text{I}_{\sigma_1}^{\alpha_1} \varphi(s)(\sigma_1). \end{aligned}$$

Since $\mathfrak{u}(\sigma_1^{-}) = \mathfrak{u}(\sigma_1^{+}) - Z_1 \mathfrak{u}(\sigma_1^{-})$, we get

$$\begin{aligned} \mathfrak{u}'(\sigma_1^{-}) &= \frac{(\varphi(\sigma_1) - \varphi(0))}{\Gamma(\gamma_1)} \cdot \\ &\quad {}^{\varphi}\text{H}_{\sigma_1}^{\gamma_1} (\varphi(s)(T) - \varphi(s))^{1-\gamma_1} \cdot \\ &\quad {}^{\varphi}\text{I}_{\sigma_1}^{\alpha_1} [\varphi(s)(T) - \varphi(s)] + {}^{\varphi}\text{I}_{\sigma_1}^{\alpha_1} \varphi(s)(\sigma_1) + Z_1 \mathfrak{u}(\sigma_1^{-}). \end{aligned}$$

If $\sigma \in (\sigma_1, \sigma_2]$, then

$\mathcal{D}_{\sigma}[\sigma] \mathfrak{u}(\sigma) = \varphi(\sigma), [\sigma] = \sigma_1$, and $\mathfrak{u}(\sigma)$ is given by

$$\begin{aligned} \mathfrak{u}(\sigma) &= \& \mathfrak{u}(\sigma_1^{+}) + \\ &\quad \frac{(\varphi(\sigma) - \varphi(\sigma_1))}{\Gamma(\gamma_1)} \cdot \\ &\quad {}^{\varphi}\text{H}_{\sigma_1}^{\gamma_1} (\varphi(s)(T) - \varphi(s))^{1-\gamma_1} \cdot \\ &\quad {}^{\varphi}\text{I}_{\sigma_1}^{\alpha_1} [\varphi(s)(T) - \varphi(s)] + {}^{\varphi}\text{I}_{\sigma_1}^{\alpha_1} \varphi(s)(\sigma_1) \\ &\quad \boxed{\text{Loading [MathJax]/jax/output/SVG/autoload/mtable.js}} \end{aligned}$$

$$\begin{aligned}
& (T) - \varphi(\sigma_0))^{1-\gamma} \bigg[\\
& w - \mathcal{I}_{\sigma_0^+}^{\gamma} \varphi(y), \varphi \\
& \varphi(s)(T) \bigg] + \frac{(\varphi(\sigma_0) - \varphi(\sigma_0))^{1-\gamma}}{w} \\
& \{ (\varphi(T) - \varphi(\sigma_0)))^{1-\gamma} \\
& ^{w - \mathcal{I}_{\sigma_0^+}^{\gamma} \varphi(y), \varphi} \\
& \varphi(s)(T) \bigg] \quad \& \\
& \{ + \mathcal{I}_{\sigma_0^+}^{\gamma} \varphi(y), \varphi \\
& \varphi(s)(\sigma_0) + \mathcal{I}_{\sigma_0^+}^{\gamma} \varphi \\
& \varphi(s)(\sigma_0) + Z_1 \mathfrak{u}(\sigma_0^{-}) \\
& \bigg). \end{aligned}$$

This means that

$$\begin{aligned}
& \begin{aligned}
& \mathfrak{u}(\sigma_0^{-}) \bigg(\\
& - \varphi(\sigma_2) \bigg) = & \frac{(\varphi(\sigma_1) - \varphi(\sigma_0))^{1-\gamma}}{w} \\
& \varphi(T) - \varphi(\sigma_0))^{1-\gamma} \bigg(\\
& w - \mathcal{I}_{\sigma_0^+}^{\gamma} \varphi(y), \varphi \\
& \varphi(s)(T) \bigg) \quad \& \\
& \{ + \frac{(\varphi(T) - \varphi(\sigma_0))^{1-\gamma}}{w} \\
& ^{w - \mathcal{I}_{\sigma_0^+}^{\gamma} \varphi(y), \varphi} \\
& \varphi(s)(\sigma_0) + \mathcal{I}_{\sigma_0^+}^{\gamma} \varphi \\
& \varphi(s)(\sigma_0) + Z_1 \mathfrak{u}(\sigma_0^{-}) \\
& \bigg). \end{aligned}
\end{aligned}$$

Since $\mathfrak{u}(\sigma_0^{-})$
 $= \mathfrak{u}(\sigma_2)$
 $\varphi(\sigma_2) = \varphi(\sigma_2)$

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\begin{aligned}
& \mathfrak{u} \bigl( \sigma_{-2}^{+} \bigr) = \frac{(\varphi(\sigma_1) - \varphi(o))^{-1} (\varphi(T) - \varphi(o))^{-1} \bigl[ w \mathcal{I}_{o^{+}}^{\gamma-1} \mathfrak{y}, \varphi(s)(T) \bigr] + \frac{(\varphi(\sigma_2) - \varphi(\sigma_1))^{-1} (\varphi(T) - \varphi(\sigma_1))^{-1} \bigl[ w \mathcal{I}_{\sigma_1^{+}}^{\gamma-1} \mathfrak{y}, \varphi(s)(\sigma_1) \bigr] + \mathcal{I}_{\sigma_1^{+}}^{\gamma-1} \mathfrak{y} \varphi(s)(\sigma_2) + Z_1 \mathfrak{u} \bigl( \sigma_{-1}^{-} \bigr) + Z_2 \mathfrak{u} \bigl( \sigma_{-2}^{-} \bigr)}{w \mathcal{I}_{o^{+}}^{\gamma-1} \mathfrak{y}, \varphi(s)(\sigma_2) + Z_1 \mathfrak{u} \bigl( \sigma_{-1}^{-} \bigr) + Z_2 \mathfrak{u} \bigl( \sigma_{-2}^{-} \bigr)}. \end{aligned}
```

If $\sigma \in (\sigma_2, \sigma_3]$, then

$\mathcal{D}_{\sigma} [\sigma]^{y,p} \varphi(\sigma) = \varphi(s)(\sigma)$, [$\sigma] = \sigma_2$, and $\mathfrak{u}(\sigma)$ is given by

```
\begin{aligned}
& \mathfrak{u}(\sigma) \\
&= \mathfrak{u} \bigl( \sigma_{-2}^{+} \bigr) + \frac{(\varphi(\sigma) - \varphi(\sigma_2))^{-1} (\varphi(T) - \varphi(\sigma_2))^{-1} \bigl[ w \mathcal{I}_{\sigma_2^{+}}^{\gamma-1} \mathfrak{y}, \varphi(s)(T) \bigr] + \mathcal{I}_{\sigma_2^{+}}^{\gamma-1} \mathfrak{y} \varphi(s)(\sigma) - \frac{(\varphi(\sigma_1) - \varphi(o))^{-1} (\varphi(T) - \varphi(o))^{-1} \bigl[ w \mathcal{I}_{o^{+}}^{\gamma-1} \mathfrak{y}, \varphi(s)(\sigma_1) \bigr] - Z_1 \mathfrak{u} \bigl( \sigma_{-1}^{-} \bigr) - Z_2 \mathfrak{u} \bigl( \sigma_{-2}^{-} \bigr)}{w \mathcal{I}_{\sigma_2^{+}}^{\gamma-1} \mathfrak{y}, \varphi(s)(\sigma_2) - \frac{(\varphi(\sigma_1) - \varphi(o))^{-1} (\varphi(T) - \varphi(o))^{-1} \bigl[ w \mathcal{I}_{o^{+}}^{\gamma-1} \mathfrak{y}, \varphi(s)(\sigma_1) \bigr] - Z_1 \mathfrak{u} \bigl( \sigma_{-1}^{-} \bigr) - Z_2 \mathfrak{u} \bigl( \sigma_{-2}^{-} \bigr)} \end{aligned}
```

$$\begin{aligned}
& \varphi(s)(T) \bigr] \&+ \frac{(\varphi(\sigma_2) - \varphi(\sigma_1))}{\Gamma(\gamma-1)} \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(T) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_1^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_2^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_1^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_2^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& + Z_1 \mathfrak{I}^{\sigma_1} u \bigr(\sigma_1^{\prime\prime} - \bigr. \\
& \bigr. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \bigr) \\
& + Z_2 \mathfrak{I}^{\sigma_2} u \bigr(\sigma_2^{\prime\prime} - \bigr. \\
& \bigr. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \bigr). \end{aligned}$$

\end{aligned}

This means that

$$\begin{aligned}
& \begin{aligned}
& \mathfrak{I}^{\sigma_3} u \bigr(\sigma_3^{\prime\prime} - \bigr. \\
& \bigr. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_1) - \varphi(0)) \bigr) = & \frac{(\varphi(\sigma_2) - \varphi(\sigma_1))}{\Gamma(\gamma-1)} \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(T) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_1^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_2^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_1^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& \left. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \right. {}^{\gamma} \mathfrak{I}^{-1} \bigr[\\
& w - \mathcal{I}_{\sigma_1}^{\sigma_2} \left(\varphi_2^{\prime\prime}(y) \right) \varphi(s)(T) \bigr] \& \\
& + Z_1 \mathfrak{I}^{\sigma_1} u \bigr(\sigma_1^{\prime\prime} - \bigr. \\
& \bigr. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \bigr) \\
& + Z_2 \mathfrak{I}^{\sigma_2} u \bigr(\sigma_2^{\prime\prime} - \bigr. \\
& \bigr. {}^{\gamma} \mathfrak{I}^{-1} (\varphi(\sigma_2) - \varphi(\sigma_1)) \bigr). \end{aligned}
\end{aligned}$$

\end{aligned}

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\mathfrak{y}, \varphi } \varpi(s)(T) \bigr] \\ &
{} + \mathcal{I}_{o^{+}}^{\circ} \mathfrak{y}, \varphi } \varpi(s)(\sigma_1) + \mathcal{I}_{\sigma_1^{+}}^{\circ} \mathfrak{y}, \varphi } \varpi(s)(\sigma_2) + \mathcal{I}_{\sigma_2^{+}}^{\circ} \mathfrak{y}, \varphi } \varpi(s)(\sigma_3) \\ & + Z_1 \mathfrak{u} \bigl( \sigma_1^{-} \\
\bigr) + Z_2 \mathfrak{u} \bigl( \sigma_2^{-} \bigr) \bigr). \end{aligned}

```

After impulse ($\mathfrak{u}(\sigma_3^{-}) = \mathfrak{u}(\sigma_3^{+}) - Z_3$)
 $\mathfrak{u}(\sigma_3^{-}) = \mathfrak{u}(\sigma_3^{+}) - Z_3$, we get

$$\begin{aligned} \begin{aligned} \mathfrak{u}(\sigma_3^{-}) &= \frac{(\varphi(\sigma_1) - \varphi(o))^{(\gamma-1)} ((\varphi(T) - \varphi(o))^{(\gamma-1)} w - \mathcal{I}_{o^+}^{\gamma-1})}{(\varphi(\sigma_2) - \varphi(\sigma_1))^{(\gamma-1)} (\varphi(T) - \varphi(\sigma_1))^{(\gamma-1)} w - \mathcal{I}_{\sigma_1^+}^{\gamma-1}} \\ &+ \frac{(\varphi(\sigma_3) - \varphi(\sigma_2))^{(\gamma-1)} ((\varphi(T) - \varphi(\sigma_2))^{(\gamma-1)} w - \mathcal{I}_{\sigma_2^+}^{\gamma-1})}{(\varphi(\sigma_3) - \varphi(\sigma_1))^{(\gamma-1)} (\varphi(T) - \varphi(\sigma_1))^{(\gamma-1)} w - \mathcal{I}_{\sigma_1^+}^{\gamma-1}} \\ &+ \frac{(\varphi(\sigma_3) - \varphi(o))^{(\gamma-1)} ((\varphi(T) - \varphi(o))^{(\gamma-1)} w - \mathcal{I}_{o^+}^{\gamma-1})}{(\varphi(\sigma_3) - \varphi(\sigma_2))^{(\gamma-1)} (\varphi(T) - \varphi(\sigma_2))^{(\gamma-1)} w - \mathcal{I}_{\sigma_2^+}^{\gamma-1}} \\ &+ \frac{(\varphi(\sigma_3) - \varphi(o))^{(\gamma-1)} ((\varphi(T) - \varphi(o))^{(\gamma-1)} w - \mathcal{I}_{o^+}^{\gamma-1})}{(\varphi(\sigma_3) - \varphi(\sigma_1))^{(\gamma-1)} (\varphi(T) - \varphi(\sigma_1))^{(\gamma-1)} w - \mathcal{I}_{\sigma_1^+}^{\gamma-1}} \end{aligned} \end{aligned}$$

$$\begin{aligned} & \text{\backslash bigr) + Z_3 \mathfrak{u} \bigl(\sigma_3^{\{-}} \\ & \text{\backslash bigr). \end{aligned}$$

If $\sigma \in (\sigma_3, \sigma_4]$, then
 $\mathcal{D}_\sigma [\sigma]^{(\mathfrak{y}, p, \varphi)} \mathfrak{u}(\sigma) = \varphi(\sigma), [\sigma] = \sigma_3$, and $\mathfrak{u}(\sigma)$
is given by

$$\begin{aligned} & \begin{aligned} & \text{\begin{aligned} & \mathfrak{u}(\sigma) \\ & = \& \mathfrak{u} \bigl(\sigma_3^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & + \frac{(\varphi(\sigma) - \varphi(\sigma_3))}{\mathfrak{y}(\sigma_3)} \bigl(\varphi(T) - \varphi(\sigma_3) \bigr) \bigr)^{\mathfrak{y}(\sigma_3)} \bigl[w - \mathcal{I}_{\sigma} \sigma_3^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & \bigr] + \mathcal{I}_{\sigma} \sigma_3^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & - \frac{\varphi(\sigma) - \varphi(\sigma_1)}{\mathfrak{y}(\sigma_1)} \bigl(\varphi(T) - \varphi(\sigma_1) \bigr) \bigr)^{\mathfrak{y}(\sigma_1)} \bigl[w - \mathcal{I}_{\sigma} \sigma_1^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & \bigr] + \frac{\varphi(\sigma) - \varphi(\sigma_2)}{\mathfrak{y}(\sigma_2)} \bigl(\varphi(T) - \varphi(\sigma_2) \bigr) \bigr)^{\mathfrak{y}(\sigma_2)} \bigl[w - \mathcal{I}_{\sigma} \sigma_2^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & \bigr] \end{aligned} \\ & \text{\backslash bigr] + \mathcal{I}_{\sigma} \sigma_3^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) } \\ & \text{\backslash bigr) \& \frac{(\varphi(\sigma) - \varphi(\sigma_1))}{\mathfrak{y}(\sigma_1)} \bigl(\varphi(T) - \varphi(\sigma_1) \bigr) \bigr)^{\mathfrak{y}(\sigma_1)} \bigl[w - \mathcal{I}_{\sigma} \sigma_1^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & \bigr] + \frac{\varphi(\sigma) - \varphi(\sigma_2)}{\mathfrak{y}(\sigma_2)} \bigl(\varphi(T) - \varphi(\sigma_2) \bigr) \bigr)^{\mathfrak{y}(\sigma_2)} \bigl[w - \mathcal{I}_{\sigma} \sigma_2^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & \bigr] \end{aligned} \\ & \text{\backslash bigr] + \mathcal{I}_{\sigma} \sigma_3^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) } \\ & \text{\backslash bigr) \& \frac{(\varphi(\sigma) - \varphi(\sigma_1))}{\mathfrak{y}(\sigma_1)} \bigl(\varphi(T) - \varphi(\sigma_1) \bigr) \bigr)^{\mathfrak{y}(\sigma_1)} \bigl[w - \mathcal{I}_{\sigma} \sigma_1^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & \bigr] + \frac{\varphi(\sigma) - \varphi(\sigma_2)}{\mathfrak{y}(\sigma_2)} \bigl(\varphi(T) - \varphi(\sigma_2) \bigr) \bigr)^{\mathfrak{y}(\sigma_2)} \bigl[w - \mathcal{I}_{\sigma} \sigma_2^{+} \bigl(\mathfrak{y}, \varphi \bigr) \varphi(s(T)) \\ & \bigr] \end{aligned} \end{math}$$

$$\begin{aligned} & \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{T}) \backslash \mathrm{bigr] } \backslash \& \\ & \{ } + \mathcal{I}_{\mathrm{o}^{+}}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \\ & \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}_1) + \mathcal{I}_{\mathrm{\sigma}}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) \\ & \mathrm{\sigma}_1^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}_2) + \mathcal{I}_{\mathrm{\sigma}_2}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}_3) \\ & + \mathcal{I}_{\mathrm{\sigma}_3}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}) \backslash \& \\ & \{ } + Z_1 \mathrm{mathfrak{u}} \backslash \mathrm{bigl} (\mathrm{\sigma}_1^{\{-} \mathrm{bigr)} + Z_2 \mathrm{mathfrak{u}} \backslash \mathrm{bigl} (\mathrm{\sigma}_2^{\{-} \mathrm{bigr)} + Z_3 \mathrm{mathfrak{u}} \backslash \mathrm{bigl} (\mathrm{\sigma}_3^{\{-} \mathrm{bigr)}. \backslash \mathrm{end}\{aligned} \end{aligned}$$

Assume that

$$\begin{aligned} & \backslash \mathrm{begin}\{aligned} & \mathrm{mathfrak{u}} \backslash \mathrm{bigl} (\mathrm{\sigma}_k^{\{+}\}^{\{+} \mathrm{bigr)} = & \& \mathrm{frac} \{ (\backslash \mathrm{varphi} (\mathrm{\sigma}_1) - \\ & \backslash \mathrm{varphi} (\mathrm{o}))^{\{ \mathrm{mathfrak{gamma}}_{-1}\}} \{ (\backslash \mathrm{varphi} \\ & (\mathrm{T}) - \backslash \mathrm{varphi} (\mathrm{o}))^{\{ \mathrm{mathfrak{gamma}}_{-1}\}} \backslash \mathrm{bigl}[\\ & w - \mathcal{I}_{\mathrm{o}^{+}}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \\ & \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{T}) \backslash \mathrm{bigr]} \backslash \& \{ } + \mathrm{frac} \{ (\backslash \mathrm{varphi} \\ & (\mathrm{\sigma}_2) - \backslash \mathrm{varphi} (\mathrm{\sigma}_1)) \\ & ^{\{ \mathrm{mathfrak{gamma}}_{-1}\}} \{ (\backslash \mathrm{varphi} (\mathrm{T}) - \backslash \mathrm{varphi} \\ & (\mathrm{\sigma}_1))^{\{ \mathrm{mathfrak{gamma}}_{-1}\}} \backslash \mathrm{bigl}[\\ & w - \mathcal{I}_{\mathrm{\sigma}_1}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{T}) \backslash \mathrm{bigr]} \backslash \& \\ & \{ } + \cdots + \mathrm{frac} \{ (\backslash \mathrm{varphi} (\mathrm{\sigma}_k) - \backslash \mathrm{varphi} \\ & (\mathrm{\sigma}_{k-1}))^{\{ \mathrm{mathfrak{gamma}}_{-1}\}} \{ (\\ & \backslash \mathrm{varphi} (\mathrm{T}) - \backslash \mathrm{varphi} (\mathrm{\sigma}_{k-1})) \\ & ^{\{ \mathrm{mathfrak{gamma}}_{-1}\}} \backslash \mathrm{bigl}[w - \\ & \mathcal{I}_{\mathrm{\sigma}_{k-1}}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{T}) \backslash \mathrm{bigr]} \backslash \& \\ & \{ } + \mathcal{I}_{\mathrm{o}^{+}}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \\ & \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}_1) + \mathcal{I}_{\mathrm{\sigma}}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \\ & \mathrm{\sigma}_1^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}_2) + \mathcal{I}_{\mathrm{\sigma}_2}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \\ & \mathrm{\sigma}_2^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}_3) + \mathcal{I}_{\mathrm{\sigma}_3}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \\ & \mathrm{\sigma}_3^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}) \backslash \& \\ & \{ } + Z_1 \mathrm{mathfrak{u}} \backslash \mathrm{bigl} (\mathrm{\sigma}_1^{\{-} \mathrm{bigr)} + Z_2 \mathrm{mathfrak{u}} \backslash \mathrm{bigl} (\mathrm{\sigma}_2^{\{-} \mathrm{bigr)} + Z_3 \mathrm{mathfrak{u}} \backslash \mathrm{bigl} (\mathrm{\sigma}_3^{\{-} \mathrm{bigr)}. \backslash \mathrm{end}\{aligned} \end{aligned}$$

Loading [MathJax]/jax/output/SVG/autoload/mtable.js	$\mathrm{\sigma}_{k-1}^{\{+}\}^{\{+} \{ \mathrm{mathfrak{y}}, \backslash \mathrm{varphi} \} \backslash \mathrm{varpi} (\mathrm{s}) (\mathrm{\sigma}_k)$
---	---

$$\begin{aligned} & \backslash sigma_k)) \backslash\& \\ & \{ } + Z_{1} \backslash mathfrak{u} \backslash bigl(\backslash sigma_{1}^{-} \backslash bigr) + Z_{2} \backslash mathfrak{u} \backslash bigl(\backslash sigma_{2}^{-} \backslash bigr) + \cdots + Z_k \backslash mathfrak{u} \backslash bigl(\backslash sigma_{k}^{-} \backslash bigr). \backslash end{aligned}$$

Then, inductively, for $\sigma \in (\sigma_k, \sigma_{k+1}]$, we have $\mathcal{D}_{\sigma} [\sigma] = \frac{(\varphi(\sigma) - \varphi(\sigma_k))^{\gamma-1}}{\Gamma(\gamma-1)} \left[w - \mathcal{I}_{\sigma}^{\gamma} \sigma \right]$, and $\mathfrak{u}(\sigma)$ is given by

$$\begin{aligned} & \begin{aligned} & \backslash begin{aligned} & \mathfrak{u}(\sigma) \\ & = & \& \mathfrak{u} \backslash bigl(\sigma_k^{+} \backslash bigr) + \\ & & \backslash frac{ (\varphi(\sigma) - \varphi(\sigma_k)) }{ \Gamma(\gamma-1) } \left[\varphi(T) - \varphi(\sigma_k) \right] \\ & & \left[w - \mathcal{I}_{\sigma}^{\gamma} \sigma \right] \\ & & \sigma_k^{+} \backslash ^{+} \backslash ^{+} \left[\mathfrak{y}, \varphi \right] \varphi(s)(T) \\ & & \backslash bigr] + \mathcal{I}_{\sigma}^{\gamma} \sigma \\ & & \sigma_k^{+} \backslash ^{+} \backslash ^{+} \left[\mathfrak{y}, \varphi \right] \varphi(s)(\sigma) \\ & & \backslash \backslash = & \sum_{i=1}^{k+1} \frac{ (\varphi(\sigma_i) - \varphi(\sigma_{i-1})) }{ \Gamma(\gamma-1) } \left[\varphi(T) - \varphi(\sigma_{i-1}) \right] \\ & & \left[w - \mathcal{I}_{\sigma}^{\gamma} \sigma \right] \\ & & \sigma_k^{+} \backslash ^{+} \backslash ^{+} \left[\mathfrak{y}, \varphi \right] \varphi(s)(T) \backslash bigr] \backslash \& \sum_{i=1}^{k+1} \left[\mathfrak{y}, \varphi \right] \varphi(s)(\sigma_i) \\ & & + \mathcal{I}_{\sigma}^{\gamma} \sigma \\ & & \sigma_k^{+} \backslash ^{+} \backslash ^{+} \left[\mathfrak{y}, \varphi \right] \varphi(s)(\sigma) \\ & & + \sum_{i=1}^{k+1} Z_i \mathfrak{u} \backslash bigl(\sigma_i^{-} \backslash bigr). \backslash end{aligned} \end{aligned}$$

Thus (a) is satisfied

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Conversely, assume that \mathfrak{u} satisfies equation (2.2).

Case 1: $\sigma \in [0, \sigma_1]$.

Replacing σ by T in (2.2), we get $\mathfrak{u}(T) = w$. On the other hand, applying

$\mathcal{D}_0^+ \{\mathfrak{u}(\sigma)\} = \mathfrak{u}'(\sigma)$; φ to both sides of (2.2) and using Lemma 2.3, we get

$$\begin{aligned} & \mathcal{D}_0^+ \{\mathfrak{u}(\sigma)\} = \mathfrak{u}'(\sigma) \\ & = \mathcal{D}_0^+ \{\mathfrak{u}(\sigma)\} - \mathfrak{u}'(y); \varphi(\sigma). \end{aligned}$$

(2.5)

Since $\mathfrak{u} \in \mathcal{PC}(J)$,

definition of $\mathcal{PC}(J)$ we have

$\mathcal{D}_0^+ \{\mathfrak{u}(\sigma)\} = \mathfrak{u}'(\sigma)$; φ $\mathfrak{u} \in \mathcal{PC}(J)$. So, (2.5) implies

$$\begin{aligned} & \mathcal{D}_0^+ \{\mathfrak{u}(\sigma)\} = \mathfrak{u}'(\sigma) \\ & = \mathcal{D}_0^+ \{\mathfrak{u}(\sigma)\} - \mathfrak{u}'(y); \varphi(\sigma) \in \mathcal{PC}(J). \end{aligned}$$

For $\varphi \in \mathcal{PC}(J)$, it is obvious that

$\mathcal{I}_0^+ \{\mathfrak{u}'(\sigma)\} = \mathfrak{u}(\sigma) - \mathfrak{u}(y); \varphi \in \mathcal{PC}(J)$. Hence ϖ and $\mathcal{I}_0^+ \{\mathfrak{u}'(\sigma)\} = \mathfrak{u}(\sigma) - \mathfrak{u}(y); \varphi$ satisfy the conditions

of Theorem 2.6. Now applying

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$\mathfrak{y})$; φ) to both sides of (2.5) and

using Theorem 2.6, we get

$$\begin{aligned} {}^H\mathcal{D}_0^{+}(\mathfrak{y}, \mathfrak{p}; \varphi) \\ } \mathfrak{u}(\sigma) = \varphi(\sigma) - \\ \frac{{}^H\mathcal{I}_0^{+}(\mathfrak{y}, \mathfrak{p})(1 - \mathfrak{y})}{(\mathfrak{p}(1 - \mathfrak{y}))} \bigg|_{\sigma=0} \Gamma \\ (\sigma) - \varphi(0) \bigg|^{\mathfrak{p}(1 - \mathfrak{y})-1}. \end{aligned}$$

(2.6)

By Theorem 2.4 we have ${}^H\mathcal{I}_0^{+}(\mathfrak{y}, \mathfrak{p})(1 - \mathfrak{y})$; $\varphi(0) = 0$. Hence (2.6) becomes

$$\begin{aligned} {}^H\mathcal{D}_0^{+}(\mathfrak{y}, \mathfrak{p}; \varphi) \\ } \mathfrak{u}(\sigma) = \varphi(\sigma), \quad \sigma \in J. \end{aligned}$$

Case 2: $\sigma \in (\sigma_k, \sigma_{k+1}]$.

By the same technique as in case 1 we can easily

prove case 2. \square

Lemma 2.10

Let γ be such that $\mathfrak{y} \in (0, 1)$, $\mathfrak{p} \in [0, 1]$, and let $f, g: J \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. If (u, ϑ) satisfies problem (1.1), then by Lemma 2.9, (u, ϑ) satisfies the following integral equations:

$$\begin{aligned} u(\sigma) = & \sum_{0 < \sigma_k < \sigma} \\ & \text{Loading [MathJax]/jax/output/SVG/autoload/mtable.js} \end{aligned}$$

$$\begin{aligned}
& (\sigma_{k-1}))^{\gamma} \{ \varphi(T) - \varphi(\sigma_{k-1}) \} \{ \\
& \varphi(T) - \varphi(\sigma_{k-1})) \\
& {}^{\gamma} \{ \varphi(\sigma_{k-1}) \} [w_1 - \\
& \mathcal{I}_{\sigma_{k-1}}^{\gamma} \{ \varphi(y) \varphi f(s, \varphi u) \\
& (s), \mathcal{D}_{\sigma_{k-1}}^{\gamma} [\varphi] \\
& {}^{\gamma} \{ \varphi(p, \varphi) \} \vartheta(s)(T)] \\
& \quad + \sum_{0 < \sigma_k < \sigma} \\
& \mathcal{I}_{\sigma_k}^{\gamma} \{ \varphi(\sigma_{k-1}) \} {}^{\gamma} \{ \\
& \varphi(y) \varphi f(s, \varphi u) \\
& (s), \mathcal{D}_{\sigma_k}^{\gamma} [\varphi] \\
& {}^{\gamma} \{ \varphi(p, \varphi) \} \vartheta(s)(\sigma_k) \\
& + \mathcal{I}_{\sigma_k}^{\gamma} \{ \varphi(\sigma_{k-1}) \} \vartheta(s)(\sigma_k) \\
& {}^{\gamma} \{ \varphi(p, \varphi) \} \vartheta(s)(\sigma_k) \\
& \quad + \sum_{0 < \sigma_k < \sigma} \\
& Z_k \mathfrak{u}(\sigma_k) (\sigma_k \{ \varphi \} \{ - \}), \quad \text{quad} \\
& \sigma \in (\sigma_k, \sigma_{k+1}] \\
& , k=1, \dots, m, \end{cases} \displaystyle \vartheta(\sigma) = \text{textstyle} \begin{cases} \sum_{0 < \sigma_k < \sigma} \frac{(\varphi(\sigma_k) - \varphi(\sigma_{k-1}))}{\sigma_k - \sigma_{k-1}} \end{cases} \\
& {}^{\gamma} \{ \varphi(\sigma_{k-1}) \} \{ (\varphi(T) - \varphi(\sigma_{k-1})) \\
& {}^{\gamma} \{ \varphi(\sigma_{k-1}) \} \{ (\varphi(T) - \varphi(\sigma_{k-1})) \\
& {}^{\gamma} \{ \varphi(\sigma_{k-1}) \} [w_2 - \\
& \mathcal{I}_{\sigma_{k-1}}^{\gamma} \{ \varphi(y) \varphi f(s, \varphi u) \\
& (s), \vartheta(s)(T)] \\
& \quad + \sum_{0 < \sigma_k < \sigma} \\
& \mathcal{I}_{\sigma_k}^{\gamma} \{ \varphi(\sigma_{k-1}) \} {}^{\gamma} \{ \\
& \varphi(y) \varphi f(s, \varphi u) \\
& (s), \mathcal{D}_{\sigma_k}^{\gamma} [\varphi] \} \vartheta(s)(\sigma_k) \\
& {}^{\gamma} \{ \varphi(p, \varphi) \} \vartheta(s)(\sigma_k) \\
& \quad + \sum_{0 < \sigma_k < \sigma} \\
& Z_k \mathfrak{u}(\sigma_k) (\sigma_k \{ \varphi \} \{ - \}), \quad \text{quad} \\
& \vartheta(s)(\sigma_k) = \text{textstyle} \begin{cases} \sum_{0 < \sigma_l < \sigma_k} \frac{(\varphi(\sigma_l) - \varphi(\sigma_{l-1}))}{\sigma_l - \sigma_{l-1}} \end{cases} \end{cases}
\end{aligned}$$

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$$\begin{aligned} & {}_k\{+}\}^{\{ \mathfrak{y}, \varphi \\ } \} g(s, \mathcal{D}_{\sigma} [\sigma] \\ & {}^{\{ \mathfrak{y}, p, \varphi \}} \mathfrak{u} \\ & (s), \vartheta(s) (\sigma) \quad \text{quad} \{ \} + \sum_{o<} \\ & \sigma_k < \sigma \} Z_k \mathfrak{u}(\sigma) \\ & _k^{\{ -\}}, \quad \sigma \in (\sigma_k, \sigma_{k+1}] , k=1, \dots, m. \end{cases} \end{aligned}$$

Consider the continuous operator Ξ

$$\begin{aligned} & : \mathcal{B} \rightarrow \mathcal{B} \text{ defined by} \\ & \Xi(u, \vartheta)(\sigma) = \bigl(\Xi_1(u, \vartheta)(\sigma), \Xi_2(\vartheta, u)(\sigma) \bigr), \end{aligned}$$

(2.7)

where

$$\begin{aligned} & \Xi_1(u, \vartheta)(\sigma) = \\ & \text{textstyle} \begin{cases} \sum_{o<} \sigma_k < \sigma \} \frac{(\varphi(\sigma_k) - \varphi(\sigma_{k-1}))^{|\gamma|-1}}{\Gamma(\gamma)} \{ (\\ & \varphi(T) - \varphi(\sigma_{k-1})) \\ & ^{|\gamma|-1} [w_1 - \\ & \mathcal{I}_{\sigma_k}^{|\gamma|} \\ & _1^{\{ +\}} \{ \mathfrak{y}, \varphi \} f(s, \mathfrak{u} \\ & (s), \mathcal{D}_{\sigma} [\sigma] \\ &)^{\{ \mathfrak{y}, p, \varphi \}} \vartheta(s)(T)] \quad \text{quad} \{ \} + \sum_{o<} \\ & \sigma_k < \sigma \} \mathcal{I}_{\sigma_k}^{|\gamma|} \\ & \{ \mathfrak{y}, \varphi \} f(s, \mathfrak{u} \\ & (s), \mathcal{D}_{\sigma}^o \{ \mathfrak{y}, p, \varphi \} \\ &)^{\{ \mathfrak{y}, p, \varphi \}} \vartheta(s)(\sigma_k) \quad \text{quad} \{ \} \\ & + \mathcal{I}_{\sigma_k}^{|\gamma|} \end{cases} \end{aligned}$$

$\{k\}^{\{ +\}} \{ \mathfrak{y}, \varphi \}$
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$$\}^{\wedge}\{\mathfrak{y}, p, \varphi\}\}\vartheta(s))(\sigma) \\) \\ \quad \sum_{0 < \sigma_k < \sigma} \\ }Z_k\mathfrak{u}(\sigma_k^{-}) , \quad \\ \sigma \in (\sigma_k, \sigma_{k+1}] , \\ k=1, \dots, m, \end{cases}$$

(2.8)

and

$$\begin{aligned} \text{Xi}_2(\vartheta, \mathfrak{u})(\sigma) = & \text{style}\begin{cases} \sum_{0 < \sigma_k < \sigma} \\ \frac{(\varphi(\sigma_k) - \varphi(\sigma_{k-1}))^{(\gamma-1)}}{(\varphi(T) - \varphi(\sigma_{k-1}))} \\ ^{\wedge}\{\mathfrak{w}_2 - \\ \mathcal{I}_{\sigma_k}^{\sigma_{k-1}} \\ ^{+}\}^{\wedge}\{\mathfrak{y}, \varphi \\ g(s, \mathcal{D}_{[\sigma]} \\ }^{\wedge}\{\mathfrak{y}, p, \varphi\}\}\mathfrak{u} \\ (s, \vartheta(T))] \\ + \sum_{0 < \sigma_k < \sigma} \\ }Z_k\mathfrak{u}(\sigma_k^{-}) , \quad \\ \sigma \in (\sigma_k, \sigma_{k+1}] , \\ k=1, \dots, m. \end{cases} \end{aligned}$$

(2.9)

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Note that the fixed points of the operator Ξ are

solutions of problem (1.1).

3 Existence of solution

In this section, we consider a general coupled system of Hilfer FDEs (1.1) involving an arbitrary function φ . To demonstrate our main results, we introduce the following hypotheses.

(H₁) The functions $f, g: J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and there exist constant numbers $\varrho_f, \varrho_g, \varrho_f' > 0$ such that for all $(u, \vartheta), (\widehat{u}, \widehat{\vartheta}) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} & \left| f(\sigma, u(\sigma), \vartheta(\sigma)) - \widehat{f}(\sigma, \widehat{u}(\sigma), \widehat{\vartheta}(\sigma)) \right| \leq \varrho_f |u(\sigma) - \widehat{u}(\sigma)| + \varrho_f' |u'(\sigma) - \widehat{u}'(\sigma)|, \\ & \left| g(\sigma, u(\sigma), \vartheta(\sigma)) - \widehat{g}(\sigma, \widehat{u}(\sigma), \widehat{\vartheta}(\sigma)) \right| \leq \varrho_g |u(\sigma) - \widehat{u}(\sigma)| + \varrho_g' |u'(\sigma) - \widehat{u}'(\sigma)|. \end{aligned}$$

(H₂) The functions $f, g: J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

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matrices multiplication are

continuous functions such that for each (

$\mathfrak{u}, \vartheta \in \mathbb{R}$, there exist nondecreasing continuous linear functions $\omega_f, \omega_g: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} & \begin{aligned} & \left| \mathfrak{u}(\sigma), \vartheta(\sigma) \right| \leq \omega_f(\sigma) + \omega_g(\sigma) \\ & \left| \mathfrak{u}'(\sigma) \right| \leq \omega_f(\sigma) + \omega_g(\sigma) \\ & \left| g(\sigma, \mathfrak{u}(\sigma), \vartheta(\sigma)) \right| \leq \omega_f(\sigma) + \omega_g(\sigma) \\ & \left| \mathfrak{u}'(\sigma) \right| \leq \omega_f(\sigma) + \omega_g(\sigma) \\ & \left| g(\sigma, \mathfrak{u}(\sigma), \vartheta(\sigma)) \right| \leq \omega_f(\sigma) + \omega_g(\sigma) \end{aligned} \end{aligned}$$

(H₃) The functions $Z_k: \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and there exists a constant $L_Z > 0$ such that

$$\begin{aligned} & \left| Z_k(\Theta) - Z_k(\Theta^*) \right| \leq L_Z |\Theta - \Theta^*|, \quad k=1, \dots, m, \\ & \Theta, \Theta^* \in \mathbb{R}. \end{aligned}$$

In the following, we will apply the Theorem 2.7 to obtain an existence result for system (1.1).

Theorem 3.1

Assume that (H₁)–(H₃) hold. If

$$\begin{aligned} Q_1 := & \frac{1}{2\Gamma(\alpha+1)} \left[\omega_f(1+\omega_g) + \omega_g^{\prime}(1+\omega_g) \right] \\ & \times \left[\omega_f^{\prime}(1+\omega_g) \omega_g \right] \left(\frac{1-\omega_f^{\prime}(1+\omega_g)}{\omega_g} \right)^{\alpha} \end{aligned}$$

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then problem (1.1) has at least one solution on J .

Proof

Define the closed ball set

$$\begin{aligned} \mathbb{B}_R = & \left\{ (\mathfrak{u}, \vartheta) \in \mathcal{B} : \right. \\ & \left. \left| \mathcal{P}(\mathfrak{u}, \vartheta) \right| \leq R, \text{ and } \right. \\ & \left. \left| \mathcal{P}(J) \right| \leq R, \text{ and } \right. \\ & \left. \left| \mathfrak{u} \right| \leq \frac{R}{2}, \text{ and } \right. \\ & \left. \left| \mathcal{P}(J) \right| \leq \frac{R}{2} \right\} \end{aligned}$$

with

$$R \geq \frac{m}{\sqrt{w_1} + \sqrt{w_2}} (1 - Q_1).$$

We will prove that the operator Ξ defined by (2.7) has a fixed point by using Theorem 2.7. For this, we divide the proof into three steps.

Step 1: $\Xi(\mathbb{B}_R) \subset \mathbb{B}_R$.

For any $(\mathfrak{u}, \vartheta) \in \mathbb{B}_R$, we have

$$\begin{aligned} & \left| \mathcal{P}(\mathfrak{u}, \vartheta) \right| \leq \left| \mathcal{P}(\mathfrak{u}, \vartheta) - \mathcal{P}(0, 0) \right| + \left| \mathcal{P}(0, 0) \right| \\ & \leq \left| \mathfrak{u} \right| + \left| \mathcal{P}(J) \right| \\ & \leq \frac{R}{2} + \frac{R}{2} = R. \end{aligned}$$

From equation (2.8) we have

$$\begin{aligned} & \left| \mathcal{P}(\mathfrak{u}, \vartheta) \right| \leq \\ & \sum_{0 < \sigma_k < \sigma} \frac{\varphi(\sigma_k - \varphi(\sigma_{k-1}))}{\sigma_k - \sigma_{k-1}} \end{aligned}$$

$$\begin{aligned}
& (\sigma_{k-1}))^{\gamma} \mathfrak{I}_{-1} \backslash \& \\
& \bigl[\left| w_1 \right| + \mathcal{I}_{\sigma_{k-1}}^{\gamma} \mathfrak{y} \varphi \bigr] \left| \mathfrak{u} \right| \\
& f(s, \mathfrak{u}(s), \mathcal{D}_{\sigma} [\sigma] \\
&)^{\gamma} \mathfrak{y} p \varphi \} \vartheta(s) \bigr] \\
& \bigr| \left(T \right) \bigr| \& + \sum_{0 < \sigma_k < \sigma} \\
& \mathcal{I}_{\sigma_k}^{\gamma} \mathfrak{y} \varphi \} \mathcal{I}_{\sigma_k}^{\gamma} \mathfrak{y} \varphi \bigr| \left. \right| \\
& f(s, \mathfrak{u}(s), \mathcal{D}_o^{\gamma} \\
&)^{\gamma} \mathfrak{y} p \varphi \} \vartheta(s) \bigr) \\
& \bigr| \left(\sigma_k \right) \bigr| \& \\
& + \mathcal{I}_{\sigma_k}^{\gamma} \mathfrak{y} \\
& \varphi \} \mathfrak{y} \varphi \bigr| \left. \right| f \\
& \bigr| \left(s, \mathfrak{u}(s), \mathcal{D}_{\sigma} [\sigma] \\
&)^{\gamma} \mathfrak{y} p \varphi \} \vartheta(s) \bigr) \\
& \bigr| \left(\sigma \right) \bigr| \& + \sum_{0 < \sigma_k < \sigma} \\
& \mathfrak{u} \bigr| \left(\sigma_k \right) \\
& \mathfrak{u} \bigr| \left(\sigma_k \right) \bigr| \leq m \left| w_1 \right| \\
& + m \mathcal{I}_{\sigma_k}^{\gamma} \mathfrak{y} \\
& \varphi \} \mathfrak{y} \varphi \bigr| \left. \right| \\
& f(s, \mathfrak{u}(s), \mathcal{D}_{\sigma} [\sigma] \\
&)^{\gamma} \mathfrak{y} p \varphi \} \vartheta(s) \bigr) \\
& \bigr| \left(T \right) \bigr| \& + m \mathcal{I}_{\sigma_k}^{\gamma} \mathfrak{y} \\
& \varphi \} \mathfrak{y} \varphi \bigr| \left. \right| f \\
& \bigr| \left(s, \mathfrak{u}(s), \mathcal{D}_{\sigma} [\sigma] \\
&)^{\gamma} \mathfrak{y} p \varphi \} \vartheta(s) \bigr) \\
& \bigr| \left(\sigma \right) \bigr| \& \\
& + \mathcal{I}_{\sigma_k}^{\gamma} \mathfrak{y} \\
& \varphi \} \mathfrak{y} \varphi \bigr| \left. \right| f \\
& \bigr| \left(s, \mathfrak{u}(s), \mathcal{D}_{\sigma} [\sigma] \\
&)^{\gamma} \mathfrak{y} p \varphi \} \vartheta(s) \bigr) \\
& \bigr| \left(\sigma \right) \bigr| + mL_Z \bigr| \left. \right| \\
& \mathfrak{u}(\sigma) \bigr| \left(\sigma \right) \bigr| \leq m \left| w_1 \right|
\end{aligned}$$

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$$\begin{aligned}
& \left| \mathfrak{u} \right| \leq \left| \mathcal{P}_C(J) \right| + \omega_f^{\prime} \omega_g^{\prime} \\
& \left| \vartheta \right| \leq \left| \mathcal{P}_C(J) \right| \\
& \left\{ \Gamma(y+1) (1-\omega_f^{\prime}) \omega_g^{\prime} \right\} \left[\varphi(T) - \varphi(0) \right] + m_L Z \left| \mathfrak{u} \right| \leq \\
& + (2m+1) \left[\frac{\omega_f^{\prime} \omega_g^{\prime}}{\Gamma(y+1) (1-\omega_f^{\prime}) \omega_g^{\prime}} \right] R \\
& \left\{ 2 \Gamma(y+1) (1-\omega_f^{\prime}) \omega_g^{\prime} \right\} \left[\varphi(T) - \varphi(0) \right] + m_L Z \frac{R}{2}.
\end{aligned}$$

Using the same technique, we get

$$\begin{aligned}
& \left| \mathfrak{u}_2 \right| \leq \left| \vartheta \right| + (2m+1) \left[\frac{\omega_f^{\prime} \omega_g^{\prime}}{\Gamma(y+1) (1-\omega_f^{\prime}) \omega_g^{\prime}} \right] R \\
& \left| \vartheta \right| \leq \left| \mathcal{P}_C(J) \right| \leq m \left| \mathfrak{u} \right| + \left| \mathfrak{u}_2 \right| + (2m+1) \left[\frac{\omega_f^{\prime} \omega_g^{\prime}}{\Gamma(y+1) (1-\omega_f^{\prime}) \omega_g^{\prime}} \right] R \\
& \left| \mathfrak{u}_2 \right| \leq \left| \mathfrak{u} \right| + m_L Z \frac{R}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left| \mathfrak{u} \right| \leq \left| \mathfrak{u}_1 \right| + \left| \mathfrak{u}_2 \right| + m_L Z \frac{R}{2} \\
& \left| \mathfrak{u}_1 \right| \leq \left| \mathfrak{u} \right| + \left| \mathfrak{u}_2 \right| + m_L Z \frac{R}{2} \\
& \left| \mathfrak{u}_2 \right| \leq \left| \mathfrak{u} \right| + m_L Z \frac{R}{2}.
\end{aligned}$$

Hence $\{X_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence.

\mathbb{B}_{\mathrm{R}}.

Step 2: E is continuous and compact.

Let $(\frac{u_n}{v_n})$ be a sequence such that $(\frac{u_n}{v_n}) \rightarrow (\frac{u}{v})$ in \mathbb{R} . Then we have

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$$\begin{aligned}
& {}_{-k}^{\{k\}^{\{-\}} \text{bigr}} - Z_{-k} \mathfrak{u} \bigl(\sigma \\
& {}_{-k}^{\{k\}^{\{-\}} \text{bigr}} \bigr) \text{vert} \quad \& \quad \leq \\
& m \mathcal{I}_{-\kappa} \sigma_k \\
& {}_1^{\{+}\}^{\{ \mathfrak{y}, \varphi \}} \bigl| \sigma \bigr| \text{vert} \bigl[f \\
& \bigl| \sigma \bigr| \bigr] {}_{n(s)} \mathcal{D}_{-\kappa} \bigl[\\
& \sigma \bigr] {}^{\{ \mathfrak{y}, p, \varphi \}} \vartheta \\
& {}_{n(s)} \text{bigr} - f \bigl| \sigma \bigr| \bigr) \\
& \mathcal{D}_{-\kappa} \bigl[\sigma \bigr] {}^{\{ \mathfrak{y}, p, \varphi \}} \vartheta \\
& \bigr] \text{vert} (T) \quad \& \\
& qquad \{ } + m \mathcal{I}_{-\kappa} \sigma_k \\
& {}_1^{\{+}\}^{\{ \mathfrak{y}, \varphi \}} \bigl| \sigma \bigr| \text{vert} \bigl[\\
& f \bigl| \sigma \bigr| \bigr] {}_{n(s)} \mathcal{D}_{-\kappa} \bigl[\\
& \sigma \bigr] {}^{\{ \mathfrak{y}, p, \varphi \}} \vartheta \\
& {}_{n(s)} \text{bigr} - f \bigl| \sigma \bigr| \bigr) \\
& \mathcal{D}_{-\kappa} \bigl[\sigma \bigr] {}^{\{ \mathfrak{y}, p, \varphi \}} \vartheta \\
& \bigr] \text{vert} (\sigma_k) \\
& \quad \& \quad qquad \{ } + \mathcal{I}_{-\kappa} \sigma_k \\
& {}_k^{\{+}\}^{\{ \mathfrak{y}, \varphi \}} \bigl| \sigma \bigr| \text{vert} \\
& \bigl| \sigma \bigr| \bigr] {}^{\{ \mathfrak{y}, p, \varphi \}} \vartheta \\
& {}_{n(s)} \text{bigr} - f \bigl| \sigma \bigr| \bigr) \\
& \mathcal{D}_{-\kappa} \bigl[\sigma \bigr] {}^{\{ \mathfrak{y}, p, \varphi \}} \vartheta \\
& \bigr] \text{vert} (\sigma) \\
& \quad \& \quad qquad \{ } + mL_Z \bigl| \sigma \bigr| \text{vert} \bigl[\\
& \mathfrak{u}_n(\sigma) - \mathfrak{u}(\sigma) \\
& \bigr] \text{vert} \quad \& \quad \leq (2m+1) \\
& \frac{\rho_f}{\rho_u} \text{Vert} \mathfrak{u}_n - \\
& \mathfrak{u} \text{Vert} \mathcal{P}(J) + \rho_\theta \\
& {}_f^{\{ \prime } \rho_g^{\{ \prime } \text{Vert} \\
& \vartheta_n - \vartheta \text{Vert} \mathcal{P}(J) \\
& \} \} \Gamma(\mathfrak{y}+1) (1 - \rho_\theta \\
& {}_f^{\{ \prime } \rho_g^{\{ \prime } \text{Vert} (\varphi(T) - \\
& \varphi(o) \bigr) {}^{\{ \mathfrak{y} \}} \quad \& \quad qquad \\
& \{ } + mL_Z \text{Vert} \mathfrak{u}_n - \mathfrak{u}
\end{aligned}$$

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By the same technique we get

$$\begin{aligned}
 & \begin{aligned} & \left\| \mathfrak{u}_n - \vartheta_n \right\|_2 (\sigma) - \left\| \mathfrak{u}_n - \vartheta_n \right\|_2 (\mathfrak{u}, \vartheta) (\sigma) \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) (\sigma) \\
 & \quad \frac{\rho_g}{\rho_f} \left\| \mathfrak{u}_n - \mathfrak{u} \right\|_2 + \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \\
 & \quad \left\| \vartheta_n - \vartheta \right\|_2 + \rho_g^{\prime \prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left\| \vartheta_n - \vartheta \right\|_2 + \rho_g^{\prime \prime \prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left\| \vartheta_n - \vartheta \right\|_2 \end{aligned} \\
 & \quad \leq (1 - \rho_f^{\prime})^{\prime} \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \\
 & \quad \leq (1 - \rho_f^{\prime})^{\prime} \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \\
 & \quad \leq \frac{\rho_g}{\rho_f} \left\| \mathfrak{u}_n - \mathfrak{u} \right\|_2 + \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left\| \mathfrak{u} - \vartheta \right\|_2 \\
 & \quad \leq \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left(\rho_g \left\| \mathfrak{u}_n - \mathfrak{u} \right\|_2 + \left\| \mathfrak{u} - \vartheta \right\|_2 \right) \\
 & \quad \leq \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left(\rho_g \left\| \mathfrak{u}_n - \mathfrak{u} \right\|_2 + \rho_g \left\| \vartheta_n - \vartheta \right\|_2 \right) \\
 & \quad \leq \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left(\rho_g \left\| \mathfrak{u}_n - \mathfrak{u} \right\|_2 + \rho_g \left\| \vartheta_n - \vartheta \right\|_2 \right) \\
 & \quad \leq \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left(\rho_g \left\| \mathfrak{u}_n - \mathfrak{u} \right\|_2 + \rho_g \left\| \vartheta_n - \vartheta \right\|_2 \right) \\
 & \quad \leq \rho_g^{\prime} \left(\Gamma(\mathcal{P}_C(J)) \right) \left(\rho_g \left\| \mathfrak{u}_n - \mathfrak{u} \right\|_2 + \rho_g \left\| \vartheta_n - \vartheta \right\|_2 \right)
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \left\| \mathfrak{u}_n - \vartheta_n \right\|_2 (\sigma) - \left\| \mathfrak{u}_n - \vartheta_n \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\sigma) \\
 & \quad \left\| \mathfrak{u} - \vartheta \right\|_2 (\mathfrak{u}, \vartheta)
 \end{aligned}$$

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```
\vartheta \Vert_{\{\mathcal{PC}\}(J)} \bigr] \&
\quad \rightarrow \quad \text{as } \\
\mathfrak{u}_n, \vartheta_n) \rightarrow ( \\
\mathfrak{u}, \vartheta) . \end{aligned}
```

Hence Ξ is continuous. Also, the operator Ξ is bounded on \mathbb{B}_R . Thus Ξ is uniformly bounded on \mathbb{B}_R . Next, we prove that Ξ is equicontinuous. Let $\sigma_1, \sigma_2 \in J$ be such that $\sigma_1 < \sigma_2$. In view of (H_2) , fixing $\sup_{(\sigma, (\mathfrak{u}, \vartheta))} \inf_{(\sigma_1, (\mathfrak{u}, \vartheta))} f(\sigma, \mathfrak{u}, \vartheta) = \widehat{f}(\sigma)$ and $\sup_{(\sigma, (\mathfrak{u}, \vartheta))} g(\sigma, \mathfrak{u}, \vartheta) = \widehat{g}(\sigma)$, we have

```
\begin{aligned}
& \left| \Xi_1 (\mathfrak{u}(\sigma_2), \vartheta(\sigma_2)) - \Xi_1 (\mathfrak{u}(\sigma_1), \vartheta(\sigma_1)) \right| \\
&= \left| \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\cdot)] - \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\sigma_1)] + \int_{\sigma_1}^{\sigma_2} \mathfrak{u}'(t) dt \right| \\
&\leq \left| \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\cdot)] - \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\sigma_1)] \right| + \left| \int_{\sigma_1}^{\sigma_2} \mathfrak{u}'(t) dt \right| \\
&\leq \left| \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\cdot)] - \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\sigma_1)] \right| + \sigma_2 - \sigma_1 \\
&\leq \left| \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\cdot)] - \mathcal{I}_{\sigma_1}^{\sigma_2} [\mathfrak{u}(\sigma_1)] \right| + \widehat{f}(\sigma_2) (\sigma_2 - \sigma_1)
\end{aligned}
```

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(\mathfrak{y})) \biggr]. \end{aligned}

(3.1)

From (3.1) we have

$$\begin{aligned} & \left| \int_{\sigma_2}^{\sigma_1} \left(\frac{\partial}{\partial \sigma} u(\sigma_2) - u(\sigma_1) \right) \vartheta(\sigma_2) d\sigma_2 \right| \\ & \leq \left| \int_{\sigma_2}^{\sigma_1} u(\sigma_2) (\sigma_2 - \sigma_1) \vartheta(\sigma_2) d\sigma_2 \right| + \left| \int_{\sigma_2}^{\sigma_1} u(\sigma_1) (\sigma_2 - \sigma_1) \vartheta(\sigma_2) d\sigma_2 \right| \\ & \leq \|u\|_{\mathcal{PC}} \|J\| \|\vartheta\|_{\mathcal{PC}} \|u\|_{\mathcal{PC}} (\sigma_1 - \sigma_2). \end{aligned}$$

(3.2)

By the same technique we get

$$\begin{aligned} & \left| \int_{\sigma_2}^{\sigma_1} \left(\frac{\partial}{\partial \sigma} u(\sigma_2) - u(\sigma_1) \right) \vartheta(\sigma_2) d\sigma_2 \right| \\ & \leq \left| \int_{\sigma_2}^{\sigma_1} u(\sigma_2) (\sigma_2 - \sigma_1) \vartheta(\sigma_2) d\sigma_2 \right| + \left| \int_{\sigma_2}^{\sigma_1} u(\sigma_1) (\sigma_2 - \sigma_1) \vartheta(\sigma_2) d\sigma_2 \right| \\ & \leq \|u\|_{\mathcal{PC}} \|J\| \|\vartheta\|_{\mathcal{PC}} \|u\|_{\mathcal{PC}} (\sigma_1 - \sigma_2). \end{aligned}$$

(3.3a)

It follows from (3.2) and (3.3a) that

$$\begin{aligned} & \left| \int_{\sigma_2}^{\sigma_1} \left(\frac{\partial}{\partial \sigma} u(\sigma_2) - u(\sigma_1) \right) \vartheta(\sigma_2) d\sigma_2 \right| \\ & \leq \left| \int_{\sigma_2}^{\sigma_1} u(\sigma_2) (\sigma_2 - \sigma_1) \vartheta(\sigma_2) d\sigma_2 \right| + \left| \int_{\sigma_2}^{\sigma_1} u(\sigma_1) (\sigma_2 - \sigma_1) \vartheta(\sigma_2) d\sigma_2 \right| \\ & \leq \|u\|_{\mathcal{PC}} \|J\| \|\vartheta\|_{\mathcal{PC}} \|u\|_{\mathcal{PC}} (\sigma_1 - \sigma_2). \end{aligned}$$

Hence Ξ is equicontinuous. By the Arzelà–Ascoli theorem we infer that Ξ is compact in \mathbb{B}_R . Therefore from the above steps we conclude that Ξ is completely continuous.

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Step 3: The set $\text{digamma} = \{\langle$

$\mathfrak{u}, \vartheta \rangle \in \mathcal{B}: (\mathfrak{u}, \vartheta) = \langle \xi, X_i \rangle$
 $\mathfrak{u}, \vartheta \rangle, \text{text}\{\} \xi \in (0,1) \}$ is bounded.

Let $\langle \mathfrak{u}, \vartheta \rangle \in \text{digamma}$. Then

$\langle \mathfrak{u}, \vartheta \rangle = \langle \xi, X_i \rangle$
 $\langle \mathfrak{u}, \vartheta \rangle$. Now, for $\sigma \in J$, we have $\mathfrak{u}(\sigma) = \xi X_{i_1} (\mathfrak{u}, \vartheta)$ and $\vartheta(\sigma) = \xi X_i (\mathfrak{u}, \vartheta)$. According to our hypotheses, we attain

$$\begin{aligned} & \begin{aligned} & \bigl\| \varphi(\sigma) - \varphi(0) \bigr\|^{1-\gamma} \\ & \bigl\| \mathfrak{u}(\sigma) \bigr\| = \| \xi X_{i_1} (\mathfrak{u}, \vartheta) \| \\ & \leq \bigl\| \varphi(\sigma) \bigr\| + \bigl\| \varphi(0) \bigr\| + \bigl\| \mathfrak{u}(\sigma) \bigr\| \end{aligned} \end{aligned}$$

By step 1 we have

$$\begin{aligned} & \begin{aligned} & \bigl\| \mathcal{PC}(J) \bigr\| = \bigl\| \varphi(\sigma) \bigr\| + \bigl\| \varphi(0) \bigr\| + \bigl\| \mathfrak{u}(\sigma) \bigr\| \\ & \leq \bigl\| \varphi(\sigma) \bigr\| + \bigl\| \varphi(0) \bigr\| + \bigl\| \mathfrak{u}(\sigma) \bigr\| \\ & + (2m+1) \biggl[\frac{\omega_f + \omega_{f'}^{\prime\prime}}{2\Gamma(y+1)(1-\omega_f)} \\ & \times \omega_{f'}^{\prime\prime} \biggr] \bigg\| \varphi(\sigma) \bigg\|^{1/2} \\ & \leq \bigl\| \varphi(\sigma) \bigr\| + \bigl\| \varphi(0) \bigr\| + \bigl\| \mathfrak{u}(\sigma) \bigr\| \end{aligned} \end{aligned}$$

(3.4)

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and

$$\begin{aligned}
 & \begin{aligned} \text{\&\&\&}\backslash\text{\Vert}\backslash\text{\vartheta}\backslash\text{\Vert} \\ & _\{\mathcal{P}C\}(J) \leq m \|\vartheta\|_2 + \\ & (2m+1) \left[\frac{\omega_f \omega_g + (\omega_g)^{\prime} R}{\Gamma(\mathfrak{y}+1)} (1-\omega_f^{\prime}) R \right]^{2\Gamma} \\ & (\omega_g \left(\frac{1}{\omega_f} + mL_Z \right) \frac{R}{\Gamma(\mathfrak{y}+1)})^{2\Gamma}. \end{aligned} \\
 & \end{aligned}$$

(3.5)

From (3.4) and (3.5) we have

$$\begin{aligned}
 & \left\| \mathcal{P}C(u, \vartheta) \right\| = \left\| \mathfrak{u} \right\| + \left\| \vartheta \right\| \\
 & \leq \left\| \mathcal{P}C(J) \right\| + \left\| \vartheta \right\| \leq R.
 \end{aligned}$$

Hence the set F is bounded. According to the above steps, together with Theorem 2.7, we conclude that Ξ has at least one fixed point. Consequently, system (1.1) has at least one solution on J . \square

In the following theorem, we prove the uniqueness of solutions to system (1.1) by using Theorem 2.8.

Theorem 3.2

Assume that (H_1) – (H_3) hold. If

$$\mathcal{Q} = (2m+1) \rho + mL_Z < 1,$$

where $\rho = \max \{ \rho_1, \rho_2 \}$ with

$$\begin{aligned}
 & \begin{aligned} \text{\&\&\&}\backslash\rho_1=\frac{\varrho_f}{\Gamma(\mathfrak{y}+1)}(1+\varrho_g)\Gamma(1-\varrho_f^{\prime})\varrho_g, \\\& \rho_2=\frac{\varrho_g^{\prime}}{\Gamma(\mathfrak{y}+1)}(1+\varrho_f)\varrho_f^{\prime}. \end{aligned} \\
 & \end{aligned}$$

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\varrho_{\{f\}}^{\prime}(\varrho_{\{g\}}),  
\end{aligned}
```

then system (1.1) has a unique solution.

Proof

Consider the closed ball \mathbb{B}_R defined in Theorem 3.1. First, we show that Ξ (\mathbb{B}_R) $\subset \mathbb{B}_R$. By the first step in Theorem 3.1 we have Ξ (\mathbb{B}_R) $\subset \mathbb{B}_R$. Next, we need to prove that Ξ is a contraction map.

Indeed, for (u, ϑ) ,

$(\widehat{u}, \widehat{\vartheta}) \in \mathbb{B}_R$ and $\sigma \in J$, we obtain

$$\begin{aligned} & \|\widehat{u} - \widehat{\vartheta}\| \leq \sum_{k=0}^{\sigma} \left(\frac{\varphi(\sigma_k - \varphi(\sigma_{k-1}))}{\varphi(\sigma_k)} \right)^{1-\frac{1}{\gamma}} \|y\|_{\varphi} \\ & \quad + \left(\frac{\varphi(\sigma - \varphi(\sigma_{k-1}))}{\varphi(\sigma)} \right)^{1-\frac{1}{\gamma}} \|y\|_{\varphi} \\ & \quad + \left(\frac{\varphi(\sigma_k - \varphi(\sigma_{k-1}))}{\varphi(\sigma_k)} \right)^{1-\frac{1}{\gamma}} \|D^\alpha u\|_{\varphi} \\ & \quad + \left(\frac{\varphi(\sigma - \varphi(\sigma_{k-1}))}{\varphi(\sigma)} \right)^{1-\frac{1}{\gamma}} \|D^\alpha \vartheta\|_{\varphi} \\ & \quad + \left(\frac{\varphi(\sigma_k - \varphi(\sigma_{k-1}))}{\varphi(\sigma_k)} \right)^{1-\frac{1}{\gamma}} \|f(s, u, D^\alpha u)\|_{\varphi} \\ & \quad + \left(\frac{\varphi(\sigma - \varphi(\sigma_{k-1}))}{\varphi(\sigma)} \right)^{1-\frac{1}{\gamma}} \|f(s, \vartheta, D^\alpha \vartheta)\|_{\varphi} \\ & \quad + \left(\frac{\varphi(\sigma_k - \varphi(\sigma_{k-1}))}{\varphi(\sigma_k)} \right)^{1-\frac{1}{\gamma}} \|g(s, u, \vartheta)\|_{\varphi} \\ & \quad + \left(\frac{\varphi(\sigma - \varphi(\sigma_{k-1}))}{\varphi(\sigma)} \right)^{1-\frac{1}{\gamma}} \|g(s, \vartheta, \vartheta)\|_{\varphi} \end{aligned}$$

$$\begin{aligned}
& [\sigma] \}^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\vartheta}(s) \biggr) \biggr| \biggr\| (\sigma \\
& _k) \& \quad \} + \mathcal{I}_\sigma \\
& _k^{+} }^{\{\mathfrak{y}, \varphi\}} \biggr\| f \\
& \biggl(s, \frac{u(s)}{D_\sigma} \biggr) \\
& \}^{\{\mathfrak{y}, p, \varphi\}} \vartheta(s) - \\
& f(s, \widehat{u}(s), D_\sigma \\
&) \\
& [\sigma] \}^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\vartheta}(s) \biggr) \biggr\| (\sigma) \\
& \& \quad \} + \sum_{0 < \sigma_k < \sigma} \\
& \biggr\| Z_k u(\sigma_k) \\
& _k^{-} \biggr) - Z_k \\
& \widehat{u}(\sigma_k) \\
& _k^{-} \biggr) \leq \\
& m \mathcal{I}_\sigma \sigma_k - \\
& 1 \}^{\{+}\}^{\{\mathfrak{y}, \varphi\}} \biggr\| f \\
& \biggl(s, \frac{u(s)}{D_\sigma} \biggr) \\
& \}^{\{\mathfrak{y}, p, \varphi\}} \vartheta(s) - \\
& f(s, \widehat{u}(s), D_\sigma \\
&) \\
& [\sigma] \}^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\vartheta}(s) \biggr) \biggr\| (T) \\
& \& \quad \} + m \mathcal{I}_\sigma \sigma_k - \\
& 1 \}^{\{+}\}^{\{\mathfrak{y}, \varphi\}} \biggr\| f \\
& \biggl(s, \frac{u(s)}{D_\sigma} \biggr) \\
& \}^{\{\mathfrak{y}, p, \varphi\}} \vartheta(s) - \\
& f(s, \widehat{u}(s), D_\sigma \\
&) \\
& [\sigma] \}^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\vartheta}(s) \biggr) \biggr\| (\\
& \sigma_k) \& \quad \} + \mathcal{I}_\sigma \\
& _k^{+} }^{\{\mathfrak{y}, \varphi\}} \biggr\| \\
& \biggl(s, \frac{u(s)}{D_\sigma} \biggr) \\
& \}^{\{\mathfrak{y}, p, \varphi\}} \vartheta(s) - \\
& f(s, \widehat{u}(s), D_\sigma \\
&) \\
& [\sigma] \}^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\vartheta}(s) \biggr) \biggr\| (\\
& \sigma_k) \}^{\{\mathfrak{y}, p, \varphi\}}
\end{aligned}$$

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$$\begin{aligned}
& \sigma) \& \quad \text{qquad} \{ } + m \bigl| \sigma \bigr| \\
Z_{\{k\}} & \mathfrak{u} \bigl(\sigma _{\{k\}}^{\{-\}} \bigr) - \\
Z_{\{k\}} & \widehat{\mathfrak{u}} \bigl(\sigma _{\{k\}}^{\{-\}} \bigr) \bigl| \sigma _{\{k\}}^{\{-\}} \bigr| \leq (2m+1) \\
& \biggl[\frac{(\varrho_f) \mathcal{V} \mathfrak{u} - \widehat{\mathfrak{u}}}{\mathcal{P}(J)} + \varrho_f^{\prime} \varrho_g^{\prime} \varrho_g^{\prime} \mathcal{V} \mathfrak{u} - \widehat{\mathfrak{u}} \biggr] \bigl| \mathcal{P}(J) \bigr| \Gamma \\
& \bigl((\mathfrak{y}+1) (1-\varrho_f^{\prime}) \varrho_g^{\prime} \mathcal{V} \mathfrak{u} - \widehat{\mathfrak{u}} \bigr) \bigl| \mathcal{P}(J) \bigr|, \end{aligned}$$

and, consequently, we obtain

$$\begin{aligned}
& \begin{aligned} & \bigl| \mathfrak{u}, \vartheta \bigr(\sigma) - \Xi_1 \bigr(\mathfrak{u}, \vartheta \bigr) \bigl| \mathcal{P}(J) \bigr| \end{aligned} \\
& \bigl| \widehat{\mathfrak{u}}, \widehat{\vartheta} \bigr(\mathfrak{u}, \vartheta \bigr) \bigl| \mathcal{P}(J) \bigr| \end{aligned}$$

(3.6)

$$\begin{aligned}
& \begin{aligned} & \leq (2m+1) \biggl[\frac{(\varrho_f) \mathcal{V} \mathfrak{u} - \widehat{\mathfrak{u}}}{\mathcal{P}(J)} + \varrho_f^{\prime} \varrho_g^{\prime} \varrho_g^{\prime} \mathcal{V} \mathfrak{u} - \widehat{\mathfrak{u}} \biggr] \bigl| \mathcal{P}(J) \bigr| \Gamma (\mathfrak{y}+1) \\
& (1-\varrho_f^{\prime}) \varrho_g^{\prime} \varrho_g^{\prime} \mathcal{V} \mathfrak{u} - \widehat{\mathfrak{u}} \biggr] \bigl| \mathcal{P}(J) \bigr| \end{aligned} \\
& \bigl| \mathfrak{u}, \vartheta \bigr(\sigma) - \Xi_1 \bigr(\mathfrak{u}, \vartheta \bigr) \bigl| \mathcal{P}(J) \bigr| \end{aligned}$$

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By the same way we obtain

```

\begin{aligned}& \bigl\Vert \Xi_{\{2\}} ( \\
& \mathfrak{u}, \vartheta ) ( \sigma ) - \Xi_{\{2\}} ( \\
& \widehat{\mathfrak{u}}, \widehat{\vartheta} ) \\
& \bigr\Vert _{\mathcal{PC}(J)} \quad \leq ( \\
& 2m+1 ) \biggl[ \frac{\rho_f \rho_g}{(\mathcal{P}C(J) + \rho_g')^2} \\
& \left| \mathfrak{u} - \widehat{\mathfrak{u}} \right| \\
& _{\mathcal{P}C(J) + \rho_g'} \\
& \left| \vartheta - \widehat{\vartheta} \right| \\
& _{\mathcal{P}C(J)} \biggr] \Gamma (\mathfrak{y}+1) \\
& (1 - \rho_f' \rho_g) \biggr] \\
& \bigl( \varphi(T) - \varphi(o) \bigr) \\
& ^{\wedge} \mathfrak{y} \quad + m L_Z \left| \vartheta - \widehat{\vartheta} \right| \\
& _{\mathcal{P}C(J)}. \end{aligned}

```

(3.8)

From (3.7) and (3.8) it follows that

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\widenset \vartheta \{ \vert _3 \backslash mathtt{PC} \{ J \} \}

Thus the operator Ξ is a contraction. So by

Theorem 2.8 system (1.1) has a unique solution. \square

4 Stability analysis

To state the main theorem, we need the following definitions. Let $\epsilon_i > 0$ and $\lambda_{\phi_i}: J \rightarrow [0, \infty)$ ($i=1,2$) be continuous functions. We consider the following inequalities:

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$$\begin{aligned} & \mathfrak{u}(\sigma) - f(\sigma) \\ & , \widehat{\mathfrak{u}}(\sigma), \mathcal{D}_\sigma^{\alpha} \\ & [\sigma]^{\alpha} \mathfrak{y}(\sigma, \varphi) \\ & \widehat{\vartheta}(\sigma) \big| \leq \epsilon_1, \end{aligned}$$

(4.1)

$$\begin{aligned} & \begin{aligned} & \mathcal{D}_\sigma^{\alpha} [\sigma]^{\alpha} \mathfrak{y}(\sigma, \varphi) \\ & (\sigma) - f(\sigma) \mathcal{D}_\sigma^{\alpha} \mathfrak{y}(\sigma, \varphi) \\ & \widehat{\vartheta}(\sigma) \big| \leq \epsilon_2, \end{aligned} \end{aligned}$$

(4.2)

$$\begin{aligned} & \begin{aligned} & \mathcal{D}_\sigma^{\alpha} [\sigma]^{\alpha} \mathfrak{y}(\sigma, \varphi) \\ & (\sigma) - f(\sigma) \mathcal{D}_\sigma^{\alpha} \\ & [\sigma]^{\alpha} \mathfrak{y}(\sigma, \varphi) \\ & \widehat{\vartheta}(\sigma) \big| \leq \epsilon_1 \lambda_{\varphi_1}(\sigma), \end{aligned} \end{aligned}$$

(4.3)

$$\begin{aligned} & \begin{aligned} & \mathcal{D}_\sigma^{\alpha} [\sigma]^{\alpha} \mathfrak{y}(\sigma, \varphi) \\ & (\sigma) - f(\sigma) \mathcal{D}_\sigma^{\alpha} \mathfrak{y}(\sigma, \varphi) \\ & \widehat{\vartheta}(\sigma) \big| \leq \epsilon_2 \lambda_{\varphi_2}(\sigma). \end{aligned} \end{aligned}$$

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(4.4)

Definition 4.1

([39])

System (1.1) is UH stable if there exists a real number $\mathcal{M} > 0$ such that for each $\epsilon = \max\{\epsilon_1, \epsilon_2\} > 0$, there exists a solution (

$\widehat{\mathfrak{u}}, \widehat{\vartheta}$) in \mathcal{B} of inequalities (4.1) and (4.2)

corresponding to a solution (

\mathfrak{u}, ϑ) in \mathcal{B} of system

(1.1) such that

$$\begin{aligned} & \left| \widehat{\mathfrak{u}} - \mathfrak{u} \right| + \left| \widehat{\vartheta} - \vartheta \right| \\ & \leq \mathcal{M} \epsilon, \quad \forall \kappa \in J. \end{aligned}$$

Definition 4.2

([39])

System (1.1) is UHR stable with respect to the nondecreasing function $\lambda_\phi(\sigma) = \max_{\kappa \in J} \{\lambda_{\phi_1}(\sigma), \lambda_{\phi_2}(\sigma)\}$ if there exists a real number $N > 0$ such that for each solution (

$\widehat{\mathfrak{u}}, \widehat{\vartheta}$) in \mathcal{B} of inequalities (4.3) and (4.4), there

exists a solution (\mathfrak{u}, ϑ) in

\mathcal{B} of system (1.1) such that

$$\begin{aligned} & \left| \widehat{\mathfrak{u}} - \mathfrak{u} \right| + \left| \widehat{\vartheta} - \vartheta \right| \\ & \leq N \epsilon, \quad \forall \kappa \in J. \end{aligned}$$

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Remark 4.3

A function (

$\widehat{\mathfrak{u}}, \widehat{\vartheta}$) in \mathcal{B} is a solution of inequalities (4.1) and (4.2) if and only if there exist functions

z_1, z_2 in $\mathcal{PC}(J)$ such that

(i) $\begin{array}{l} \bigl\| \begin{array}{c} c \\ \widehat{\mathfrak{u}}(\kappa) \end{array} \bigr\| \leq \epsilon_1, \text{ } \sigma \in J, \\ \|\widehat{\mathfrak{u}}(\kappa)\| \leq \epsilon_2, \text{ } \sigma \in J, \end{array}$

(ii) $\begin{array}{l} \bigl\| \begin{array}{c} \mathcal{D}_\sigma^y p(\varphi) \\ \widehat{\mathfrak{u}}(\sigma) = f(\sigma) \\ , \widehat{\mathfrak{u}}(\sigma), \mathcal{D}_\sigma^y \\ [\sigma]^y p(\varphi) \\ \widehat{\vartheta}(\sigma) + z_1(\sigma) \\ , \sigma \in J, \end{array} \bigr\| \leq \mathcal{D}_\sigma^y \\ \mathcal{D}_\sigma^y p(\varphi) \widehat{\vartheta}(\sigma) + z_2(\sigma) \\ , \sigma \in J. \end{array}$

$\begin{array}{l} \bigl\| \begin{array}{c} \mathcal{D}_\sigma^y p(\varphi) \\ \widehat{\mathfrak{u}}(\sigma) = f(\sigma) \\ , \widehat{\mathfrak{u}}(\sigma), \mathcal{D}_\sigma^y \\ [\sigma]^y p(\varphi) \\ \widehat{\vartheta}(\sigma) + z_1(\sigma) \\ , \sigma \in J, \end{array} \bigr\| \leq \mathcal{D}_\sigma^y \\ \mathcal{D}_\sigma^y p(\varphi) \widehat{\vartheta}(\sigma) + z_2(\sigma) \\ , \sigma \in J. \end{array}$

Lemma 4.4

Let $y \in (0,1)$ and $p \in [0,1]$. If a function (

$\widehat{\mathfrak{u}}, \widehat{\vartheta}$) in \mathcal{B} satisfies inequalities (4.1) and (4.2), then ($\widehat{\mathfrak{u}}, \widehat{\vartheta}$) satisfies the following integral inequalities:

$\begin{cases} \widehat{\mathfrak{u}}(\sigma) - A_{\widehat{\mathfrak{u}}}(\sigma) \\ - \widehat{\mathfrak{u}}(\sigma) - I_{\widehat{\mathfrak{u}}}(\sigma) \end{cases}$

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$$\begin{aligned} & \widehat{\mathfrak{u}}(s), \mathcal{D}_{\sigma} [\widehat{\vartheta}(s)) (\sigma) \text{ vert } \leq \epsilon_1 K, \\ & \widehat{\vartheta}(s)) (\sigma) - \mathcal{A} \widehat{\vartheta}(\sigma) - \mathcal{I}_{\sigma} \widehat{\vartheta}(\sigma) \\ & \quad + \mathfrak{k}^+ \widehat{y}(\sigma) \widehat{\varphi}(s, \mathcal{D}_{\sigma} [\widehat{u}(s))] \\ & \quad + \widehat{y}(\sigma) \widehat{\varphi}(s, \mathcal{D}_{\sigma} [\widehat{u}(s))]) \leq \epsilon_2 K, \end{aligned}$$

where

$$\begin{aligned} & \mathcal{A} \widehat{\vartheta}(s) := \sum_{0<\sigma_k<\sigma} \frac{(\varphi(\sigma_k) - \varphi(\sigma_{k-1}))}{\Gamma(\gamma_k)} \\ & \quad \times \left(\varphi(T) - \varphi(\sigma_{k-1}) \right)^{\gamma_k-1} \left[w_1 \mathcal{I}_{\sigma_k} \widehat{u}(s) \right. \\ & \quad \left. + \mathfrak{y}(\sigma_k) \widehat{\varphi}(s, \widehat{u}(s)) \right] \\ & \quad + \sum_{0<\sigma_k<\sigma} \mathcal{I}_{\sigma_k} \widehat{u}(s) \\ & \quad + \sum_{0<\sigma_k<\sigma} \mathfrak{y}(\sigma_k) \widehat{\varphi}(s, \widehat{u}(s)) \\ & \quad + \sum_{0<\sigma_k<\sigma} \mathcal{A} \widehat{\vartheta}(\sigma_k) Z_{\sigma_k} \widehat{\vartheta}(\sigma_k) \widehat{\varphi}(s, \widehat{u}(\sigma_k)) \\ & \quad = \sum_{0<\sigma_k<\sigma} \frac{(\varphi(\sigma_k) - \varphi(\sigma_{k-1}))}{\Gamma(\gamma_k)} \left(\varphi(T) - \varphi(\sigma_{k-1}) \right)^{\gamma_k-1} \\ & \quad \times \left[w_1 \mathcal{I}_{\sigma_k} \widehat{u}(s) + \mathfrak{y}(\sigma_k) \widehat{\varphi}(s, \widehat{u}(s)) \right] \\ & \quad + \sum_{0<\sigma_k<\sigma} \mathcal{I}_{\sigma_k} \widehat{u}(s) \\ & \quad + \sum_{0<\sigma_k<\sigma} \mathfrak{y}(\sigma_k) \widehat{\varphi}(s, \widehat{u}(s)) \\ & \quad + \sum_{0<\sigma_k<\sigma} \mathcal{A} \widehat{\vartheta}(\sigma_k) Z_{\sigma_k} \widehat{\vartheta}(\sigma_k) \widehat{\varphi}(s, \widehat{u}(\sigma_k)) \\ & \quad = \widehat{\vartheta}(s) \end{aligned}$$

$$\begin{aligned}
& (\sigma_{k-1}))^{\{\mathfrak{y}\}} [\\
& w_2-\mathcal{I}_{\sigma_k} \sigma_{k-1}^{\{+}})^{\{\mathfrak{y}, \varphi\}} \\
& }g(s, \mathcal{D}_{[\sigma]} \sigma_{k-1}^{\{+}}) \\
& \widehat{\mathfrak{u}}(s, \widehat{\vartheta}(s)) \\
& (s)(T)] \\
& \hphantom{\widehat{\mathfrak{u}}(s, \widehat{\vartheta}(s))} - \sum_{o < \sigma_k < \sigma} \\
& \mathcal{I}_{\sigma_k} \sigma_{k-1}^{\{+}})^{\{\mathfrak{y}, \varphi\}} g(s, \mathcal{D}_{[\sigma]} \sigma_{k-1}^{\{+}}) \\
&] \widehat{\mathfrak{u}}(s, \widehat{\vartheta}(s)) \\
& (s)(\sigma_k) - \sum_{o < \sigma_k < \sigma} Z_k \widehat{\vartheta}(\sigma_k) \\
& \sigma_k^{\{-}}), \text{end} \{ \text{cases} \}
\end{aligned}$$

and

$$K := (2m+1) \frac{(\varphi(T) - \varphi(o))}{\Gamma(\mathfrak{y}+1)}.$$

Proof

Indeed, by Remark 4.3 we have

$$\begin{aligned}
& \text{textstyle} \begin{cases} \mathcal{D}_{[\sigma]} \sigma_{k-1}^{\{+}} \widehat{\mathfrak{u}}(\sigma) = f(\sigma) \\
\widehat{\mathfrak{u}}(\sigma) = g(\sigma, \mathcal{D}_{[\sigma]} \sigma_{k-1}^{\{+}}) \widehat{\vartheta}(\sigma) + z_1(\sigma) \\
\sigma \in J, \\ \& \mathcal{D}_{[\sigma]} \sigma_{k-1}^{\{+}} \widehat{\mathfrak{u}}(\sigma) = \widehat{\mathfrak{u}}(\sigma) \end{cases} \\
& (\sigma) = g(\sigma, \mathcal{D}_{[\sigma]} \sigma_{k-1}^{\{+}}) \widehat{\vartheta}(\sigma) \\
& \sigma^{\{+}} \widehat{\mathfrak{u}}(\sigma),
\end{aligned}$$

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 $\sigma \in J$. end{cases}

Then, for $\sigma \in (\sigma_k, \sigma_{k+1}]$

, $k=1, \dots, m$, we get

$$\begin{aligned}
& \text{\textstyle\begin{cases} \text{vert} \\
\widehat{\mathfrak{u}}(\sigma) - \mathcal{A}\{ \\
\widehat{\mathfrak{u}}(\sigma) - \mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} f(s, \\
\widehat{\mathfrak{u}}(s), \mathcal{D}_0^{\alpha} \{ \mathfrak{y}, p, \varphi \}) \\
\widehat{\vartheta}(s) \} \text{vert} \\ \quad \leq \sum_{0 < \sigma_k < \sigma} \frac{(\\
\varphi(\sigma_k) - \varphi(\sigma_{k-1}))}{\mathfrak{g}(\gamma-1)} \{ (\\
\varphi(T) - \varphi(\sigma_{k-1})) \}^{\{ \mathfrak{g}(\gamma-1) \}} [\\
\mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \text{vert} z_1(s) \\
\text{vert} (T)] \\ \quad + \sum_{0 < \sigma_k < \sigma} \mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \text{vert} z_1(s) \text{vert} (\\
\sigma_k) + \mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \text{vert} z_1(s) \\
\text{vert} (\\
\sigma_k), \text{vert} \widehat{\vartheta} \\
(\sigma) - \mathcal{A}\{ \widehat{\vartheta} \\
(\sigma) - \mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} g(s, \mathcal{D}_0^{\alpha} \{ \mathfrak{y}, p, \varphi \} \\
\widehat{\mathfrak{u}}(s), \widehat{\vartheta}(s)) \} \text{vert} \\ \quad \leq \sum_{0 < \sigma_k < \sigma} \frac{(\\
\varphi(\sigma_k) - \varphi(\sigma_{k-1}))}{\mathfrak{g}(\gamma-1)} \{ (\\
\varphi(T) - \varphi(\sigma_{k-1})) \}^{\{ \mathfrak{g}(\gamma-1) \}} [\\
\mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \text{vert} z_2(s) \\
\text{vert} (T)] \\ \quad + \sum_{0 < \sigma_k < \sigma} \mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \text{vert} z_2(s) \text{vert} (\\
\sigma_k) + \mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \text{vert} z_2(s) \\
\text{vert} (\\
\sigma_k) \} \text{vert} \widehat{\vartheta} \\
(\sigma) - \mathcal{A}\{ \widehat{\vartheta} \\
(\sigma) - \mathcal{I}_{\sigma} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} \\
_{\sigma_k}^{+} \}^{\{ \mathfrak{y}, \varphi \}} g(s, \mathcal{D}_0^{\alpha} \{ \mathfrak{y}, p, \varphi \} \\
\widehat{\mathfrak{u}}(s), \widehat{\vartheta}(s)) \} \text{vert} \\
& \boxed{\text{Loading [MathJax]/jax/output/SVG/autoload/mtable.js } \sigma}
\end{aligned}$$

$$\begin{aligned} & -\{k\} + \mathcal{I}_{\sigma} \\ & -\{k\}^{+} \cdot \{\mathfrak{y}, \varphi\} \mid z_2(s) \\ & \mid (\sigma). \end{aligned}$$

It follows that

$$\begin{aligned} & \text{\\textstyle\\begin{cases} \mid \widehat{\mathfrak{u}}(\sigma) - \mathcal{A}_{\sigma} \\ \mid \widehat{\mathfrak{u}}(\sigma) - \mathcal{I}_{\sigma} \\ -\{k\}^{+} \cdot \{\mathfrak{y}, \varphi\} f(s, \\ \mid \widehat{\mathfrak{u}}(s), \mathcal{D}_{\sigma} [\sigma] \\] \cdot \{\mathfrak{y}, p, \varphi\} \mid \widehat{\vartheta}(s)) (\sigma) \mid \leq \epsilon_1 K, \\ \mid \widehat{\vartheta}(\sigma) - \mathcal{A}_{\sigma} \\ \mid \widehat{\vartheta}(\sigma) - \mathcal{I}_{\sigma} \\ -\{k\}^{+} \cdot \{\mathfrak{y}, \varphi\} \\ g(s, \mathcal{D}_{\sigma} [\sigma] \\] \cdot \{\mathfrak{y}, p, \varphi\} \\ \mid \widehat{\mathfrak{u}}(s), \mid \widehat{\vartheta}(s),) (\sigma) \mid \leq \epsilon_2 K. \end{cases}} \end{aligned}$$

□

In the forthcoming theorem, we prove the stability results for system (1.1).

Theorem 4.5

Assume that (H_{1}) and (H_{2}) hold. Then

$$\begin{aligned} & \text{\\textstyle\\begin{cases} \mathcal{D}_{\sigma} [\sigma] \\ \cdot \{\mathfrak{y}, p, \varphi\} \mid \mathfrak{u}(\sigma) \\ = f(\sigma, \mathfrak{u}(\sigma), \mathcal{D}_{\sigma} [\sigma] \\ \cdot \{\mathfrak{y}, p, \varphi\} \mid \vartheta(\sigma)), \sigma \in J, \\ \mathcal{D}_{\sigma} [\sigma] \cdot \{\mathfrak{y}, p, \varphi\} \mid \vartheta(\sigma) \\ = g(\sigma, \mathcal{D}_{\sigma} [\sigma] \cdot \{\mathfrak{y}, p, \varphi\} \mid \vartheta(\sigma)) \end{cases}} \end{aligned}$$

(4.5)

are UH stable, provided that $\Delta = (1-\Lambda_{1f})(1-\Lambda_{2g}) - \Lambda_{1g}\Lambda_{2f} \neq 0$, where

$$\begin{aligned} \Lambda_{1f} &= \frac{\varphi(T) - \varphi(0)}{\Gamma(\mathfrak{y}+1)(1-\varrho_f)}, \quad \text{where } \varrho_f \\ &= \frac{\varphi'(T) - \varphi'(0)}{\Gamma(\mathfrak{y}+1)\varrho_f\varrho_g}, \\ \Lambda_{2f} &= \frac{\varphi(T) - \varphi(0)}{\Gamma(\mathfrak{y}+1)(1-\varrho_g)}, \quad \text{where } \varrho_g \\ &= \frac{\varphi'(T) - \varphi'(0)}{\Gamma(\mathfrak{y}+1)\varrho_f\varrho_g}, \\ \Lambda_{1g} &= \frac{\varphi(T) - \varphi(0)}{\Gamma(\mathfrak{y}+1)(1-\varrho_f)}, \\ \Lambda_{2g} &= \frac{\varphi(T) - \varphi(0)}{\Gamma(\mathfrak{y}+1)(1-\varrho_g)}. \end{aligned}$$

Proof

Let $\epsilon = \max\{\epsilon_1, \epsilon_2\} > 0$, let $\widehat{u}, \widehat{\vartheta}$ be functions satisfying inequalities (4.1) and (4.2), and let (u, ϑ) be the unique solution of the following system

$$\begin{cases} \mathcal{D}_{\sigma}^{\mathfrak{y}, p} u(\sigma) = f(\sigma, u(\sigma)), \\ \mathcal{D}_{\sigma}^{\mathfrak{y}, p} \vartheta(\sigma) = g(\sigma, \vartheta(\sigma)), \end{cases}$$

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```

\sigma _{k=1,\dots,m} \mathcal{D}_{\sigma} [
\sigma ] }^{\mathfrak{y},p,\varphi} \vartheta
(\sigma )=g(\sigma ,\mathcal{D}_{\sigma} [\sigma ])^{
\mathfrak{y},p,\varphi} \mathfrak{u}(\sigma
),\vartheta (\sigma ),\quad \sigma \in J:= [ 0,T ]
,\sigma \neq \sigma _{k=1,\dots,m} \Delta
\mathfrak{u} \vert _{\sigma =\sigma _k}=
\Delta \widehat{\mathfrak{u}} \vert _{\sigma =\sigma _k}=
\sigma _k=\widehat{\mathfrak{u}} \vert _{\sigma =\sigma _k}
(\sigma _k^{-}),\quad k=1,\dots,m, \Delta
\vartheta \vert _{\sigma =\sigma _k}=\Delta
\widehat{\vartheta } \vert _{\sigma =\sigma _k}=\Delta
\widehat{\vartheta } \vert _{\sigma =\sigma _k}=\widehat{\vartheta }(\sigma
_k^{-}),\quad k=1,\dots,m, \Delta
\widehat{\mathfrak{u}}(T)=\mathfrak{u}(T)
(w_1),\quad \widehat{\vartheta }(T)=\vartheta (T)=w_2.
\end{cases}

```

Then by Theorem 3.1 we have

```

\textstyle\begin{cases} \mathfrak{u}(\sigma
)=\mathcal{A}(\mathfrak{u})+\mathcal{I}_{\sigma}-
_{k=1}^{+}\mathfrak{y},\varphi
f(s,\mathfrak{u}(s),\mathcal{D}_{\sigma} [\sigma ]
}^{\mathfrak{y},p,\varphi} \vartheta (s)) ( \sigma
), \quad \vartheta (\sigma )=\mathcal{A}(\vartheta
)+\mathcal{I}_{\sigma}-
_{k=1}^{+}\mathfrak{y},\varphi
g(s,\mathcal{D}_{\sigma} [\sigma ])^{
\mathfrak{y},p,\varphi} \mathfrak{u}(s),\vartheta
(s)) ( \sigma ).\end{cases}

```

Since

```

\textstyle\begin{cases} \Delta \mathfrak{u} \vert _{\sigma =\sigma _k}=\Delta
\widehat{\mathfrak{u}} \vert _{\sigma =\sigma _k}=\widehat{\mathfrak{u}}(\sigma
_k^{-}),\quad \widehat{\mathfrak{u}}(T)=\mathfrak{u}(T)
\end{cases}

```

```

_{k}^{-}), \quad k=1, \dots, m, \Delta \vartheta
\sigma = \sigma_k = \Delta
\widehat{\vartheta} \sigma = \sigma (\Delta
_k) = Z_k \widehat{\vartheta} (\Delta
_k), \quad k=1, \dots, m,
\widehat{u}(T) = w_1, \quad \widehat{\vartheta}(T) = \vartheta
(T) = w_2, \end{cases}

```

we can easily prove that

$\mathcal{A}u = \mathcal{A}\widehat{u}$ and $\mathcal{A}\vartheta = \mathcal{A}\widehat{\vartheta}$. Hence from (H₂) and Lemma 4.4, for each $\sigma \in J$,

we have

$$\begin{aligned}
& \begin{aligned}
& \left| \widehat{\mathcal{A}}u(\sigma) - \mathcal{A}u(\sigma) \right| \\
& = & \left| \widehat{\mathcal{A}}u(\sigma) - \mathcal{A}\widehat{u}(\sigma) \right| \\
& & - \left| \mathcal{A}\widehat{u}(\sigma) - \mathcal{A}u(\sigma) \right| \\
& & - \left| \mathcal{A}u(\sigma) - u(\sigma) \right|
\end{aligned} \\
& \leq \left| f(s, \widehat{u}(s), \mathcal{D}^\sigma [\widehat{u}]^+ (s)) \right| \\
& & + \left| f(s, \widehat{u}(s), \mathcal{D}^\sigma [\widehat{u}]^+ (s)) - f(s, \widehat{u}(s), \mathcal{D}^\sigma [u]^- (s)) \right| \\
& & + \left| f(s, \widehat{u}(s), \mathcal{D}^\sigma [u]^- (s)) - f(s, u(s), \mathcal{D}^\sigma [u]^- (s)) \right| \\
& & + \left| f(s, u(s), \mathcal{D}^\sigma [u]^- (s)) - u(\sigma) \right|
\end{aligned}$$

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$$\begin{aligned}
& (s), \mathcal{D}_{[\sigma]} \}^{\wedge} \\
& \backslash \text{mathfrak}{y,p,\varphi} \} \} \widehat{\vartheta} \\
& (s) \biggr) (\sigma) \biggr| \text{vert} \\
& & \& + \mathcal{I}_{\sigma} \\
& _k^{+} })^{\wedge} \{ \mathfrak{y}, \varphi \} \biggr| \text{vert} f \\
& \biggl(s, \widehat{\mathfrak{u}}(s), \mathcal{D}_{[\sigma]} \}^{\wedge} \{ \mathfrak{y,p,\varphi} \} \\
& \widehat{\vartheta}(s) \biggr) (\sigma) - f \biggl(s, \\
& \mathfrak{u}(s), \mathcal{D}_{[\sigma]} \}^{\wedge} \{ \mathfrak{y,p,\varphi} \} \biggr) \vartheta(s) \biggr) (\sigma) \\
& \biggr| \text{vert} \leq K \epsilon \\
& _1 } + + \mathcal{I}_{\sigma} \\
& _k^{+} })^{\wedge} \{ \mathfrak{y}, \varphi \} \biggr| \text{vert} \\
& f \biggl(s, \widehat{\mathfrak{u}}(s), \mathcal{D}_{[\sigma]} \}^{\wedge} \{ \mathfrak{y,p,\varphi} \} \\
& \widehat{\vartheta}(s) \biggr) (\sigma) - \\
& f \biggl(s, \mathfrak{u}(s), \mathcal{D}_{[\sigma]} \}^{\wedge} \{ \mathfrak{y,p,\varphi} \} \biggr) \vartheta(s) \biggr) (\sigma) \\
& \biggr| \text{vert} \end{aligned}$$

(4.6)

and

```
\begin{aligned} \bigl\| \widehat{\vartheta}(\sigma) - \widehat{\vartheta}(\sigma) \bigr\| = & \\ \bigl\| \widehat{\vartheta}(\sigma) - \mathcal{A}_{\widehat{\vartheta}}(\sigma) - \mathcal{I}_{\widehat{\vartheta}}(\sigma) & \\ - \frac{1}{k} \sum_{j=1}^k \left( \frac{\widehat{y}_j}{\widehat{\varphi}_j} \right)^{\mathcal{D}[\sigma]} \left( \frac{\widehat{u}(s_j)}{\widehat{\vartheta}(s_j)} \right) \bigr\| \\ & + \|\mathcal{I}_{\widehat{\vartheta}}(\sigma)\| \\ & + \|\mathcal{I}_{\widehat{\vartheta}}(\sigma) - \mathcal{A}_{\widehat{\vartheta}}(\sigma)\| \\ & + \|\mathcal{A}_{\widehat{\vartheta}}(\sigma) - \widehat{\vartheta}(\sigma)\| \\ & + \|\widehat{\vartheta}(\sigma) - \widehat{\vartheta}(\sigma)\| \end{aligned}
```

$$\begin{aligned}
& \sigma] }^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\mathfrak{u}}(s), \widehat{\vartheta}(s) \big| (\sigma) \big| \text{vert} \leq \big| \mathcal{A}(\sigma) - \mathcal{I}_\sigma \\
& \widehat{\vartheta}(\sigma) - \mathcal{I}_\sigma \\
& _k^{+} \{\mathfrak{y}, \varphi\} g(s, \mathcal{D}_\sigma) [\sigma] \\
& }^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\mathfrak{u}}(s), \widehat{\vartheta}(s) \big| (\sigma) \big| \text{vert} \\
& & + \mathcal{I}_\sigma \\
& _k^{+} \{\mathfrak{y}, \varphi\} \big| \text{vert} g \\
& \big| (s, \mathcal{D}_\sigma) [\sigma] \\
& }^{\{\mathfrak{y}, p, \varphi\}} \\
& \widehat{\mathfrak{u}}(s), \widehat{\vartheta}(s) \big| (\sigma) - g(s, \mathcal{D}_\sigma) [\\
& \sigma] }^{\{\mathfrak{y}, p, \varphi\}} \mathfrak{u}(s), \vartheta(s) \big| (\sigma) \big| \text{vert} \leq \\
& K \epsilon_2 + \mathcal{I}_\sigma \\
& _k^{+} \{\mathfrak{y}, \varphi\} \big| \text{vert} g(s, \mathcal{D}_\sigma) [\sigma] }^{\{\mathfrak{y}, p, \varphi\}} \widehat{\mathfrak{u}}(s), \\
& \widehat{\vartheta}(s) \big| (\sigma) - g(s, \mathcal{D}_\sigma) [\sigma] }^{\{\mathfrak{y}, p, \varphi\}} \mathfrak{u}(s), \vartheta(s) \big| (\sigma) - \\
& g(s, \mathcal{D}_\sigma) [\sigma] }^{\{\mathfrak{y}, p, \varphi\}} \mathfrak{u}(s), \vartheta(s) \big| (\sigma) - \\
& g(s, \mathcal{D}_\sigma) [\sigma] . \end{aligned}$$

(4.7)

Thus by (H₁) we have

$$\begin{aligned}
& \big| \mathfrak{u} \big| \widehat{\mathfrak{u}}(s) - \mathfrak{u}(s) \big| \text{vert} \\
& \leq K \epsilon_1 + \big| \frac{\varrho_f}{\mathcal{P}_C(J)} \big| \big| \widehat{\mathfrak{u}}(s) - \mathfrak{u}(s) \big| \big| \mathcal{P}_C(J) + \varrho_f
\end{aligned}$$

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 $\widehat{\vartheta} - \vartheta \big| \text{vert}$

$$\begin{aligned} & \left\{ \mathcal{P}C(J) \right\} \Gamma(\mathfrak{y}+1) \\ & (1-\varrho_f^{\prime} \varrho_g) \bigg] \\ & \bigl(\varphi(T) - \varphi(o) \bigr) \\ & {}^{\mathfrak{y}}. \end{aligned}$$

By the same technique we get

$$\begin{aligned} & \left| \widehat{\vartheta} - \vartheta \right| \leq K \epsilon_2 + \left[\frac{\varrho_f \varrho_g}{\varphi(T) + \varrho_g^{\prime}} \right] \\ & \left| \widehat{u} - u \right| \leq \left| \mathcal{P}C(J) \right| \Gamma(\mathfrak{y}+1) \\ & (1-\varrho_f^{\prime} \varrho_g) \bigg] \\ & \bigl(\varphi(T) - \varphi(o) \bigr) \\ & {}^{\mathfrak{y}}. \end{aligned}$$

It follows that

$$\begin{aligned} & \left| \widehat{u} - u \right| \leq \left| \mathcal{P}C(J) \right| (1 - \Lambda_{1f}) - \left| \widehat{\vartheta} - \vartheta \right| \\ & \left| \widehat{\vartheta} - \vartheta \right| \leq \left| \mathcal{P}C(J) \right| \Lambda_{1g} \leq K \epsilon_1 \end{aligned}$$

(4.8)

and

$$\begin{aligned} & \left| \widehat{u} - u \right| \leq \left| \mathcal{P}C(J) \right| (1 - \Lambda_{2g}) - \left| \widehat{\vartheta} - \vartheta \right| \\ & \left| \widehat{\vartheta} - \vartheta \right| \leq \left| \mathcal{P}C(J) \right| \Lambda_{2f} \leq K \epsilon_2. \end{aligned}$$

(4.9a)

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Inequalities (4.8) and (4.9a) can be rewritten in the matrix form

$$\begin{aligned} & \begin{pmatrix} 1 - \Lambda_{1f} & -\Lambda \\ -\Lambda_{1g} & 1 - \Lambda_{2f} & 1 - \Lambda_{2g} \end{pmatrix} \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \\ & \left\| \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \right\|_{\mathcal{PC}(J)} \leq \left\| \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \right\|_{\mathcal{PC}(J)} \\ & \leq \epsilon_1 K + \epsilon_2 K \end{aligned}$$

By simple computations this inequality becomes

$$\begin{aligned} & \left\| \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \right\|_{\mathcal{PC}(J)} \leq \frac{1}{\Delta} \begin{pmatrix} 1 - \Lambda_{2g} & -\Lambda \\ -\Lambda_{1g} & 1 - \Lambda_{2f} & 1 - \Lambda_{1f} \end{pmatrix} \begin{pmatrix} \epsilon_1 K + \epsilon_2 K \\ \epsilon_1 K + (1 - \Lambda_{1f}) \epsilon_1 K + (1 - \Lambda_{2f}) \epsilon_2 K \end{pmatrix} \end{aligned}$$

This leads to

$$\begin{aligned} & \begin{aligned} & \left\| \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \right\|_{\mathcal{PC}(J)} \leq \\ & \frac{(1 - \Lambda_{2g}) \epsilon_1 K + \Lambda_{1g} \epsilon_2 K}{\Delta}, \\ & \left\| \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \right\|_{\mathcal{PC}(J)} \leq \frac{\Lambda_{2f} \epsilon_1 K + (1 - \Lambda_{1f}) \epsilon_1 K + (1 - \Lambda_{2f}) \epsilon_2 K}{\Delta}. \end{aligned} \end{aligned}$$

Thus

$$\begin{aligned} & \begin{aligned} & \left\| \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \right\|_{\mathcal{PC}(J)} \leq \\ & \frac{(1 - \Lambda_{2g}) \epsilon_1 K + \Lambda_{1g} \epsilon_2 K}{\Delta}, \\ & \left\| \begin{pmatrix} \widehat{\mathfrak{u}} - \mathfrak{u} \\ \widehat{\vartheta} - \vartheta \end{pmatrix} \right\|_{\mathcal{PC}(J)} \leq \frac{\Lambda_{2f} \epsilon_1 K + (1 - \Lambda_{1f}) \epsilon_1 K + (1 - \Lambda_{2f}) \epsilon_2 K}{\Delta}. \end{aligned} \end{aligned}$$

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$$\begin{aligned} & \mathfrak{u} - \mathfrak{u} \mid_{\mathcal{P}C} \\ & (J) + \mathfrak{u} \mid_{\widehat{\vartheta}} - \vartheta \mid_{\mathcal{P}C} \\ & \leq \frac{2}{\Lambda_{2g} + \Lambda_{1g} + \Lambda_{2f} - \Lambda_{1f}} \Delta \epsilon K \leq \\ & \mathcal{M} \epsilon, \end{aligned}$$

(4.10)

where $\epsilon = \max \{\epsilon_1, \epsilon_2\}$ and $\mathcal{M} = \frac{2}{\Lambda_{2g} + \Lambda_{1g} + \Lambda_{2f} - \Lambda_{1f}} \Delta K$. Hence by inequality (4.10) and Definition 4.1 the solution of system (1.1) is Ulam–Hyers stable. Next, by setting $\lambda_\phi = \epsilon \mathcal{M}$ such that $\lambda_\phi \phi(0) = 0$ system (1.1) is generalized Ulam–Hyers stable. \square

5 An example

Consider the following problem:

$$\begin{aligned} & \text{cases} \mathcal{D}_{\sigma} [y] \\ & \quad \begin{cases} \mathfrak{u}(\sigma) = \frac{\varphi(\sigma) - \varphi(\frac{1}{5})}{10e^{(\varphi(\sigma) - \varphi(0))}} \left[\frac{\varphi(\sigma)}{1 + |\varphi(\sigma)|} + \frac{|\varphi(\sigma)|}{\mathcal{D}_{\sigma} [\sigma]} \right. \\ \quad \left. {}^{\sigma}y, \mathfrak{p}, \varphi \right] \mid_{\vartheta} (\sigma) \\ \quad \text{for } \sigma \in (0, 1] - \{ \frac{1}{5} \} \end{cases} \end{aligned}$$

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$$\begin{aligned}
 & (\frac{1}{5}))^{(\gamma_1)} \cdot \varphi(\sigma) - \varphi(0) \} \cdot [\frac{\partial}{\partial \sigma} \left(\varphi(\sigma) \right)^{(\gamma_1)} \\
 & \quad \cdot \mathcal{D}_{\sigma}^{\gamma_1} u(\sigma) \cdot \varphi(\sigma) \cdot \varphi'(\sigma)] + \frac{1}{5} \cdot \varphi(\sigma) \cdot \varphi'(\sigma) \\
 & \quad \cdot \mathcal{D}_{\sigma}^{\gamma_1} u(\sigma) = 0, \quad \sigma \in (0,1] \\
 & \text{with } u(0) = 3, \quad u'(0) = 2.
 \end{aligned}$$

(5.1)

Here $y = \frac{1}{3}$, $p = \frac{1}{2}$, $\gamma = \frac{2}{3}$, $w_1 = 3$, $w_2 = 2$. Set $\varphi(\sigma) = e^{\sigma}$.

Example 5.1

Define $f, g: (0,1) \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\begin{aligned}
 f(\sigma, u) &= \frac{1}{5} \cdot \varphi(\sigma) \cdot \varphi'(u) \\
 g(\sigma, u) &= \frac{1}{5} \cdot \varphi(\sigma) \cdot \varphi'(u) + \frac{1}{5} \cdot \varphi(\sigma) \cdot \varphi'(u) \cdot \varphi(u) \\
 &= \frac{1}{5} \cdot e^{\sigma} \cdot e^{\sigma} \cdot \varphi(u) + \frac{1}{5} \cdot e^{\sigma} \cdot e^{\sigma} \cdot \varphi(u) \cdot \varphi(u) \\
 &= \frac{1}{5} \cdot e^{2\sigma} \cdot \varphi(u) + \frac{1}{5} \cdot e^{2\sigma} \cdot \varphi(u) \cdot \varphi(u)
 \end{aligned}$$

$$\begin{aligned}
& \mathfrak{u}(\sigma) \Big| + \frac{\partial}{\partial \sigma} \\
& \mathcal{D}_{\sigma}[\sigma] \\
& \left(\mathfrak{y}, \mathfrak{p}, \varphi \right) \vartheta \\
& (\sigma) \Big| \left(1 + \left| \mathcal{D}_{\sigma}[\sigma] \right| \right)^2 \left(\mathfrak{y}, \mathfrak{p}, \varphi \right) \vartheta \\
& (\sigma) \Big| \bigg] , \quad \& \bigl(\sigma \\
& \mathcal{D}_{\sigma}[\sigma]^2 \left(\mathfrak{y}, \mathfrak{p}, \varphi \right) \\
& \} \mathfrak{u}(\sigma), \vartheta(\sigma) \\
&) \bigr) = \frac{\left(\varphi(\sigma) - \varphi\left(\frac{1}{5}\right) \right)^2}{10e^{\left(\varphi(\sigma) - \varphi(0) \right)}} \bigg[\frac{\partial}{\partial \sigma} \\
& \mathcal{D}_{\sigma}[\sigma] \\
& \left(\mathfrak{y}, \mathfrak{p}, \varphi \right) \\
& \} \mathfrak{u}(\sigma) \Big| \left(1 + \left| \mathcal{D}_{\sigma}[\sigma] \right| \right)^2 \left(\mathfrak{y}, \mathfrak{p}, \varphi \right) \\
& \vartheta(\sigma) \Big| \Big| \bigg] , \quad \text{end}\{aligned\}
\end{aligned}$$

and $Z_1, Z_2 : \mathbb{R} \rightarrow$

$Z_1(\mathfrak{u}) = \frac{\left| \mathfrak{u} \right|}{8(1 + \left| \mathfrak{u} \right|)}$

and

$Z_2(\vartheta) = \frac{\left| \vartheta \right|}{8(1 + \left| \vartheta \right|)}$.

Then, for $(\mathfrak{u}, \vartheta)$,

$$\begin{aligned}
& (\widehat{\mathfrak{u}}, \widehat{\vartheta}) \in \mathbb{R} \times \mathbb{R}, \text{ we have} \\
& \begin{aligned}
& \left| \widehat{\mathfrak{u}} \right| = \left| \mathfrak{u} \right|, \quad \left| \widehat{\vartheta} \right| = \left| \vartheta \right| \\
& \widehat{\mathfrak{u}} = \mathfrak{u} + \frac{\partial}{\partial \sigma} \mathfrak{u}(\sigma) \\
& \widehat{\vartheta} = \vartheta + \frac{\partial}{\partial \sigma} \vartheta(\sigma)
\end{aligned}
\end{aligned}$$

$$\begin{aligned}
& \left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\
& \left. + \left| \mathfrak{u}'(\sigma) - \widehat{\mathfrak{u}}'(\sigma) \right| + \left| \vartheta'(\sigma) - \widehat{\vartheta}'(\sigma) \right| \right) \\
& \left| \mathfrak{u}'(\sigma) - \widehat{\mathfrak{u}}'(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\
& \left. + \left| \mathfrak{u}''(\sigma) - \widehat{\mathfrak{u}}''(\sigma) \right| + \left| \vartheta''(\sigma) - \widehat{\vartheta}''(\sigma) \right| \right) \\
& \left| \mathfrak{u}''(\sigma) - \widehat{\mathfrak{u}}''(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\
& \left. + \left| \mathfrak{u}'''(\sigma) - \widehat{\mathfrak{u}}'''(\sigma) \right| + \left| \vartheta'''(\sigma) - \widehat{\vartheta}'''(\sigma) \right| \right) \\
& \left| \mathfrak{u}'''(\sigma) - \widehat{\mathfrak{u}}'''(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\
& \left. + \left| \mathfrak{u}^{(4)}(\sigma) - \widehat{\mathfrak{u}}^{(4)}(\sigma) \right| + \left| \vartheta^{(4)}(\sigma) - \widehat{\vartheta}^{(4)}(\sigma) \right| \right)
\end{aligned}$$

\begin{aligned} & \left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\ & \left. + \left| \mathfrak{u}'(\sigma) - \widehat{\mathfrak{u}}'(\sigma) \right| + \left| \vartheta'(\sigma) - \widehat{\vartheta}'(\sigma) \right| \right) \\ & \left| \mathfrak{u}'(\sigma) - \widehat{\mathfrak{u}}'(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\ & \left. + \left| \mathfrak{u}''(\sigma) - \widehat{\mathfrak{u}}''(\sigma) \right| + \left| \vartheta''(\sigma) - \widehat{\vartheta}''(\sigma) \right| \right) \\ & \left| \mathfrak{u}''(\sigma) - \widehat{\mathfrak{u}}''(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\ & \left. + \left| \mathfrak{u}'''(\sigma) - \widehat{\mathfrak{u}}'''(\sigma) \right| + \left| \vartheta'''(\sigma) - \widehat{\vartheta}'''(\sigma) \right| \right) \\ & \left| \mathfrak{u}'''(\sigma) - \widehat{\mathfrak{u}}'''(\sigma) \right| \leq \frac{1}{10} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\ & \left. + \left| \mathfrak{u}^{(4)}(\sigma) - \widehat{\mathfrak{u}}^{(4)}(\sigma) \right| + \left| \vartheta^{(4)}(\sigma) - \widehat{\vartheta}^{(4)}(\sigma) \right| \right) \end{aligned}

and

$$\begin{aligned}
& \left| Z_1(\mathfrak{u}) - Z_1(\widehat{\mathfrak{u}}) \right| \leq \frac{1}{8} \left(\left| \mathfrak{u}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| + \left| \widehat{\mathfrak{u}}(\sigma) - \widehat{\mathfrak{u}}(\sigma) \right| \right. \\
& \left. + \left| \mathfrak{u}'(\sigma) - \widehat{\mathfrak{u}}'(\sigma) \right| + \left| \mathfrak{u}''(\sigma) - \widehat{\mathfrak{u}}''(\sigma) \right| \right) \\
& \left| Z_2(\vartheta) - Z_2(\widehat{\vartheta}) \right| \leq \frac{1}{8} \left(\left| \vartheta(\sigma) - \widehat{\vartheta}(\sigma) \right| + \left| \widehat{\vartheta}(\sigma) - \widehat{\vartheta}(\sigma) \right| \right. \\
& \left. + \left| \vartheta'(\sigma) - \widehat{\vartheta}'(\sigma) \right| + \left| \vartheta''(\sigma) - \widehat{\vartheta}''(\sigma) \right| \right)
\end{aligned}$$

Here $\varrho_f = \varrho_f' = \varrho$, $\varrho_g = \varrho_g' = \omega$, $\varrho_{f'} = \omega_f = \omega$, $\varrho_{g'} = \omega_g = \frac{1}{10}$ and $L_{Z_1} = L_{Z_2} = \frac{1}{8}$. From the given data we deduce that conditions (H_1) , (H_2) , and (H_3) hold. Thus all the conditions of Theorem 3.1 are satisfied. Therefore problem (1.1) has at least one solution on $[0,1]$. Moreover, we have $\rho_1 = \rho_2 = 0,1$ and

$\boxed{\mathbf{Thus\ all\ conditions\ of}}$
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Theorem 3.2 are satisfied. Therefore problem (1.1)

has a unique solution on $[0,1]$.

Finally, for $\epsilon = \max \{\|u\|_1, \|u\|_2\} > 0$, we find that the inequalities

$$\begin{aligned} & \|\mathcal{D}_0^{\sigma} [y_p \varphi] - \widehat{\mathfrak{u}}(\sigma) f(\sigma)\| \leq \epsilon, \\ & \|\widehat{\mathfrak{u}}(\sigma) - \widehat{\vartheta}(\sigma) f(\sigma)\| \leq \epsilon, \\ & \|\mathcal{D}_0^{\sigma} [\vartheta_p \varphi] - \widehat{\mathfrak{u}}(\sigma) f(\sigma)\| \leq \epsilon, \\ & \|\widehat{\vartheta}(\sigma) - \widehat{\vartheta}(\sigma) f(\sigma)\| \leq \epsilon, \end{aligned}$$

are satisfied. Then equation (4.5) is Ulam–Hyers stable with

$$\|\widehat{\mathfrak{u}}(\sigma) - \widehat{\vartheta}(\sigma)\| \leq M \epsilon, \quad \forall \sigma \in J.$$

where

$$M = 2.3 > 0.$$

6 Concluding remarks

We obtained the existence, uniqueness, and UH stability of solutions for a new problem of φ -Hilfer FDEs with impulse conditions. Our investigations were based on the reduction of FDEs to FIEs and application the standard Leray–Schauder and

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in this paper are more general and cover many of the parallel problems that contain particular cases of the function φ , because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives; for instance, for various values of a function φ and parameter \mathfrak{p} , the coupled system (1.1) includes coupled systems of FDEs involving the Hilfer, Hadamard, Katugampola, and many other fractional derivative operators, which are described in the introduction.

Availability of data and materials

The authors declare that all data and materials in this paper are available and veritable.

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Contributions

MAA: Writing original draft, conceptualization, writing review and editing, methodology. SKP: Supervision, review, and editing. All authors read and approved the final manuscript.

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Ethics approval and consent to participate

Not applicable.

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The authors declare that they have no competing interests.

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Not applicable.

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