Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2022, Article ID 6716830, 20 pages
https://doi.org/10.1155/2022/6716830

Research Article

The Hermite–Hadamard–Mercer Type Inequalities via Generalized Proportional Fractional Integral Concerning Another Function

Tariq A. Aljaaidi [1] **and Deepak B. Pachpatte** [2]

[1]*Department of Mathematics, Hajjah University, Hajjah, Yemen*
[2]*Department of Mathematics, Dr.Babasaheb Ambedkar Marathwada University, Aurangabad (M.S) 431001, India*

Correspondence should be addressed to Tariq A. Aljaaidi; tariq10011@gmail.com

Received 17 December 2021; Accepted 29 January 2022; Published 28 March 2022

Academic Editor: Ram N. Mohapatra

In order to be able to study cosmic phenomena more accurately and broadly, it was necessary to expand the concept of calculus. In this study, we aim to introduce a new fractional Hermite–Hadamard–Mercer's inequality and its fractional integral type inequalities. To facilitate that, we use the proportional fractional integral operators of integrable functions with respect to another continuous and strictly increasing function. Moreover, we establish some new fractional weighted φ-proportional fractional integral Hermite–Hadamard–Mercer type inequalities. Furthermore, in this article, we are keen to present some special cases related to our current study compared to the previous work of the inequality under study.

1. Introduction

There is no doubt that a researcher in the field of calculus knows the significant importance that fractional calculus has acquired recently due to its multiple and important uses in many fields in the natural sciences and technology, especially in physics, fluid dynamics, biology, image processing, control theory, computer networking, and signal processing.

Fractional calculus is the generalized form of classical integrals and derivatives for the order is a noninteger, which comes within the framework of mathematicians' relentless pursuit of developing mathematics to make it more general and useable in most cases that may encounter when studying and analyzing natural phenomena. According to this, we can say fractional calculus has become the focus of a large number of researchers' attention. As a result, a lot of extensions and generalizations have appeared especially on the classical fractional calculus like the definitions of Riemann–Liouville (RL) and Caputo. Actually, the derivative Riemann–Liouville is the most general concept and the most uniform and natural. In general, there are numerous other definitions of fractional operators such as Erdélyi–Kober, Hilfer, Katugampola, Hadamard, and Riesz which are just a few examples to make reference to [1, 2]. It should be noted that there are many modern fractional operators proposed by many researchers and perhaps the most prominent of them is the recently proposed ABC operator by Atangana and Baleanu [3, 4].

Definition 1. The function $g\colon ([a,z]\subseteq\mathbb{R}) \longrightarrow \mathbb{R}$ is said to be a convex function if the inequality

$$g(\eta r + (1-\eta)s) \le \eta g(r) + (1-\eta) g(s), \tag{1}$$

holds for all $r, s \in [a,z]$ and $\eta \in [0,1]$. We say that g is a concave function if inequality (1) is reversed. In general, the real-valued function g is said to be a convex function on $[a,z]$ if and only if for all $y_1, y_2, \ldots, y_n \in [a,z]$ and for any $\eta_i \in [a,z]$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{i=n}\eta_i = 1$, we have

$$g\left(\sum_{i=1}^{n}\eta_i y_i\right) \le \sum_{i=1}^{n}\eta_i g(y_i). \tag{2}$$

This well-known inequality is called Jensen inequality [5].

Convexity of functions with their features is one of the most useful properties among other categories of functions in the important fields of applied sciences, especially statistics and mathematics, which according to its own useful definition has a geometric interpretation. Furthermore, it is a vital part of inequalities theory and has become the leading point for creating numerous inequalities such as Jensen's inequality, Hadamard's inequality with its type inequalities, and Steffensen's inequality. One of these inequalities that are closely related to the convexity of functions is the Hermite–Hadamard inequality, which has a well-known area in the space of inequalities theory. This inequality was initially proposed by Hermite in 1881, but it did not come into prominence until it was enriched by Hadamard in 1893 [6] as follows:

$$g\left(\frac{a+z}{2}\right) \leq \frac{1}{(a-z)} \int_a^z g(y)\mathrm{d}y \leq \frac{g(a)+g(z)}{2}, \quad a,z \in \mathbb{R}, a<z, \tag{3}$$

where g is a convex function on $[a,z]$, which is called the Hermite–Hadamard (H-H) inequality. Significantly, H-H inequality recently has become the focus of attention of several mathematicians and researchers due to its remarkable applications and its spacious uses in several diverse areas. Concerning that, a large number of articles have appeared that contain extensions and generalizations of this inequality (see [7–12]).

Lemma 1 (see [13]). *Suppose that the function g: $([a, z]\subseteq\mathbb{R}) \longrightarrow \mathbb{R}$ is convex on $[a,z]$. Then, we have*

$$g(a+z-y) \leq g(a) + g(z) - g(y). \tag{4}$$

In general, McD Mercer in [13] proved the following generalization of each of inequality (2) and inequality (4), which is called the well-known Jensen–Mercer inequality.

Theorem 1. *Suppose that the function g: $([a,z]\subseteq\mathbb{R}) \longrightarrow \mathbb{R}$ is convex on $[a,z]$. Then, for all $y_1, y_2, \ldots, y_n \in [a,z]$ and for any $\eta_i \in [a,z]$, $i = 1,2,\ldots,n$, with $\sum_{i=1}^{i=n}\eta_i = 1$, we have*

$$g\left(a+z-\sum_{i=1}^{n}\eta_i y_i\right) \leq g(a) + g(z) - \sum_{i=1}^{n}\eta_i g(y_i). \tag{5}$$

Inequality (5) is a matter of supreme interest due to much information and its explicit boundary conditions. In mathematics and engineering sciences, Jensen–Mercer's inequality and associated inequalities have wonderful applications and their generalizations and extensions have been an excellent topic of research for mathematicians and authors in the past few years as seen through a variety of investigations on the subject. Moradi and Furuichi [14] (2020) presented some new generalizations and improvements of Jensen–Mercer's type inequalities. Khan et al. [15] (2020) applied Jensen–Mercer's inequality in information theory to compute new ratings for Csiszár and related divergence. For more generalizations and details of Jensen–Mercer's type inequalities, see [16–18].

Many researchers, motivated by all the above literature, did a lot of research and were able to derive a new inequality which is a mixture of H-H and Jensen–Mercer's inequalities, which was named the Hermite–Hadamard–Mercer's inequality which is our focus through introducing this article.

Ogulmus and Sarikaya [19] (2019) established fractional integral Hermite–Hadamard–Mercer's inequalities for RL operators.

Theorem 2. *Let g: $[a,z] \longrightarrow \mathbb{R}$ be a convex function. The following inequalities hold:*

$$\begin{aligned} g\left(a+z-\frac{x+y}{2}\right) &\leq g(a)+g(z)-\frac{\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\mathcal{I}_{x^+}^{\beta} g(y)+\mathcal{I}_{y^-}^{\beta} g(x)\right\} \\ &\leq g(a)+g(z)-g\left(\frac{x+y}{2}\right), \end{aligned} \tag{6}$$

$$\begin{aligned} g\left(a+z-\frac{x+y}{2}\right) &\leq \frac{\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\mathcal{I}_{\{a+z-y\}^+}^{\beta} g(a+z-x) + \mathcal{I}_{\{a+z-x\}^-}^{\beta} g(a+z-y)\right\} \\ &\leq \frac{g(a+z-y)+g(a+z-x)}{2} \\ &\leq g(a)+g(z)-\frac{g(x)+g(y)}{2}. \end{aligned} \tag{7}$$

The same authors, in the same work, presented the following inequality.

Theorem 3. *Let g: $[a,z] \longrightarrow \mathbb{R}$ be a convex function. The following inequalities hold:*

$$\begin{aligned} g\left(a+z-\frac{x+y}{2}\right) &\leq \frac{\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\mathcal{I}_{\left\{a+z-\frac{x+y}{2}\right\}^-}^{\beta} g(a+z-y) + \mathcal{I}_{\left\{a+z-\frac{x+y}{2}\right\}^+}^{\beta} g(a+z-x)\right\} \\ &\leq g(a)+g(z)-\frac{g(x)+g(y)}{2}. \end{aligned} \tag{8}$$

Iscan [20] (2020) employed the RL fractional integral to investigate some weighted Hermite–Hadamard–Mercer's type inequalities as follows.

Theorem 4. *Let g: $[a,z] \longrightarrow \mathbb{R}$ be a convex differentiable function on (a,z) and w: $[a,z] \longrightarrow \mathbb{R}$ be a nonnegative integrable function. Then, the following inequalities hold:*

$$\begin{aligned} &g\left(a+z-\frac{x+y}{2}\right)\mathcal{I}^{\beta}_{\{a+z-x\}^-} w(a+z-y)\\ &\le \frac{1}{2}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{a+z-x\}^-}\mathrm{gw}(a+z-y)+\mathcal{I}^{\beta}_{\{a+z-y\}^+}\mathrm{gw}(a+z-x)\right\}\\ &\le \frac{1}{2}[g(a+z-x)+g(a+z-y)]\mathcal{I}^{\beta}_{\{a+z-x\}^-}w(a+z-y). \end{aligned} \tag{9}$$

Iscan also, in the same work, gave the following weighted Hermite–Hadamard–Mercer's inequality.

Theorem 5. *Let g: $[a,z] \longrightarrow \mathbb{R}$ be a convex differentiable function on (a,z) and w: $[a,z] \longrightarrow \mathbb{R}$ be a nonnegative integrable function. Then, the following inequalities hold:*

$$\begin{aligned} &g\left(a+z-\frac{x+y}{2}\right)\mathcal{I}^{\beta}_{\{a+z-x\}^-} w(a+z-y)\\ &\le \frac{1}{2}\left\{\mathcal{I}^{\beta}_{\{a+z-x\}^-}\mathrm{gw}(a+z-y)+\mathcal{I}^{\beta}_{\{a+z-y\}^+}\mathrm{gw}(a+z-x)\right\}\\ &\le (g(a)+g(z))\mathcal{I}^{\beta}_{\{x\}^+}w(y)-\frac{1}{2}\left\{\mathcal{I}^{\beta}_{\{x\}^+}\mathrm{gw}(y)+\mathcal{I}^{\beta}_{\{y\}^-}\mathrm{gw}(x)\right\}\\ &\le \left\{g(a)+g(z)-g\left(\frac{x+y}{2}\right)\right\}\mathcal{I}^{\beta}_{\{x\}^+}w(y). \end{aligned} \tag{10}$$

Abdeljawad et al. [21] (2020) established some inequalities of Hermite–Hadamard–Mercer type inequalities employing RL fractional integral. Butt et al. [22] (2020) proved some Hermite–Hadamard–Mercer type inequalities for convex functions by using the conformable fractional integrals. Chu et al. [23] (2020) presented some generalizations of Hermite–Hadamard–Mercer type inequalities via Katugampola fractional integral. Recently, in 2021, Vivas-Cortez et al. [24] used generalized RL to present some Hermite–Hadamard–Mercer type inequalities involving convex functions. For more recent studies and generalizations of this inequality, please see [25–28].

All of what we mentioned above prompts us to study Hermite–Hadamard–Mercer's inequality via the recently generalized operators. Here, in this study, we aim to establish Hermite–Hadamard–Mercer's inequality and its type inequalities for convex functions employing proportional fractional integral operators involving continuous strictly increasing functions. We also aim to present some fractional weighted Hermite–Hadamard–Mercer type inequalities via the current generalized integral operators. Along with this study, we are able to discuss some special cases and some relationships between our current study and previous studies.

The organization of this research paper will be as follows: In Section 2, we will mention some notations, definitions, and preparatory acquaintance which are used in this work. Section 3 is devoted to the first part of our major results which contain Hermite–Hadamard–Mercer's inequalities. Throughout Section 4, we provide the fractional weighted Hermite–Hadamard–Mercer's type inequalities.

2. Essential Preliminaries

Here, we characterize some of the basic properties and some definitions of several elementary fractional integral operators which include the final generalized fractional operator we used to obtain and discuss our new results.

Definition 2 (see [1]). Suppose that the function g is integrable on $[a,z]$ and $a \ge 0$. Then, for all $\beta > 0$, we have

$$\mathcal{I}^{\beta}_{a^+}g(y)=\frac{1}{\Gamma(\beta)}\int_a^y (y-u)^{\beta-1}g(u)\mathrm{d}u, u>a, \tag{11}$$

$$\mathcal{I}^{\beta}_{z^-}g(y)=\frac{1}{\Gamma(\beta)}\int_y^z (u-y)^{\beta-1}g(u)\mathrm{d}u, y<z, \tag{12}$$

where $\Gamma(\beta)=\int_0^{\infty}e^{-x}x^{\beta-1}\mathrm{d}x$ is the Gamma function and $\mathcal{I}^{0}_{a^+}g(y)=\mathcal{I}^{0}_{z^-}g(y)=g(y)$. The notations $\mathcal{I}^{\beta}_{a^+}g(y)$ and $\mathcal{I}^{\beta}_{z^-}g(y)$ are called, respectively, the left- and right-sided Riemann–Liouville fractional integrals of a function g for the order β.

Definition 3 (see [1, 2]). Suppose that the function g is integrable on the interval $\digamma$, and let φ be an increasing function, where $\varphi(y) \in C^1(\digamma,\mathbb{R})$ such that $\varphi'(y) \ne 0$ and $y \in \digamma$. Then, for all $\beta > 0$, we have

$$\varphi\mathcal{I}^{\beta}_{a^+}g(y)=\frac{1}{\Gamma(\beta)}\int_a^y \varphi'(u)[\varphi(y)-\varphi(u)]^{\beta-1}g(u)\mathrm{d}u, \tag{13}$$

$$\varphi\mathcal{I}^{\beta}_{z^-}g(y)=\frac{1}{\Gamma(\beta)}\int_y^z \varphi'(u)[\varphi(u)-\varphi(y)]^{\beta-1}g(u)\mathrm{d}u. \tag{14}$$

The notations $\varphi\mathcal{I}^{\beta}_{a^+}g(y)$ and $\varphi\mathcal{I}^{\beta}_{z^-}g(y)$ are, respectively, called the left- and right-sided φ-Riemann–Liouville fractional integrals of a function g for the order β.

Definition 4 (see [29]). For the function g, let $\delta > 0$, and we have for all $\beta \in \mathbb{C}$ and $\mathrm{Re}(\beta) \ge 0$,

$$\begin{aligned} \left(D^{\beta,\delta}_{a^+}g\right)(y) &= D^{m,\delta}\mathcal{I}^{m-\beta,\delta}_{a^+}g(y)\\ &=\frac{D^{m,\delta}_y}{\delta^{m-\beta}\Gamma(m-\beta)}\int_a^y \exp\left[\frac{\delta-1}{\delta}(y-u)\right]\\ &\quad\cdot (y-u)^{m-\beta-1}g(u)\mathrm{d}u, \end{aligned} \tag{15}$$

$$\begin{aligned} \left(D^{\beta,\delta}_{z^-}g\right)(y) &= \gamma D^{m,\delta}\mathcal{I}^{m-\beta,\delta}_{z^-}g(y)\\ &=\frac{\gamma D^{m,\delta}_y}{\delta^{m-\beta}\Gamma(m-\beta)}\int_y^z \exp\left[\frac{\delta-1}{\delta}(u-y)\right]\\ &\quad\cdot (u-y)^{m-\beta-1}g(u)\mathrm{d}u, \end{aligned} \tag{16}$$

where

$$D^{m,\delta} = \underbrace{D^{\delta}D^{\delta}\ldots.D^{\delta}}_{m-\text{times}}, m = [\mathrm{Re}(\beta)] + 1, \tag{17}$$

$$\begin{aligned}\left(\gamma D^{\delta} g\right)(y) &= (1-\delta)g(y) - \delta g'(y), \gamma D^{m,\delta} \\ &= \underbrace{\gamma D^{\delta}_{\gamma} D^{\delta} \ldots . \gamma D^{\delta}}_{m-\text{times}}.\end{aligned} \tag{18}$$

The notations $(D^{\beta,\delta}_{a^+} g)(y)$ and $(D^{\beta,\delta}_{z^-} g)(y)$ are, respectively, called the left- and right-sided proportional fractional derivatives of a function g for the order β.

Definition 5 (see [29]). For the integrable function g, let $\delta > 0$, and we have for all $\beta \in \mathbb{C}$ and $\mathrm{Re}(\beta) \geq 0$,

$$\begin{aligned}\left(\mathcal{I}^{\beta,\delta}_{a^+} g\right)(y) &= \frac{1}{\delta^{\beta}\Gamma(\beta)} \int_a^y \exp\left[\frac{\delta-1}{\delta}(y-u)\right] \\ &\cdot (y-u)^{\beta-1} g(u)\mathrm{d}u,\end{aligned} \tag{19}$$

$$\begin{aligned}\left(\mathcal{I}^{\beta,\delta}_{z^-} g\right)(y) &= \frac{1}{\delta^{\beta}\Gamma(\beta)} \int_y^z \exp\left[\frac{\delta-1}{\delta}(u-y)\right] \\ &\cdot (u-y)^{\beta-1} g(u)\mathrm{d}u.\end{aligned} \tag{20}$$

The notations $(\mathcal{I}^{\beta,\delta}_{a^+} g)(y)$ and $(\mathcal{I}^{\beta,\delta}_{z^-} g)(y)$ are, respectively, called the left- and right-sided proportional fractional integrals of a function g for the order β.

Definition 6 (see [30]). For the integrable function g and for the strictly increasing continuous function φ on $[a,z]$, let $\delta \in (0,1]$, and we have for all $\beta \in \mathbb{C}$ and $\mathrm{Re}(\beta) \geq 0$,

$$\begin{aligned}\left(\varphi D^{\beta,\delta}_{a^+} g\right)(y) &= \varphi D^{m,\delta\varphi} \mathcal{I}^{m-\beta,\delta}_{a^+} g(y) \\ &= \frac{\varphi D^{m,\delta}_{y}}{\delta^{m-\beta}\Gamma(m-\beta)} \int_a^y \exp\left[\frac{\delta-1}{\delta}(\varphi(y)-\varphi(u))\right] \\ &\cdot (\varphi(y)-\varphi(u))^{m-\beta-1}\varphi'(u)g(u)\mathrm{d}u,\end{aligned} \tag{21}$$

$$\begin{aligned}\left(\varphi D^{\beta,\delta}_{z^-} g\right)(y) &= {}^{\varphi}_{\gamma} D^{m,\delta\varphi} \mathcal{I}^{m-\beta,\delta}_{z^-} g(y) \\ &= \frac{{}^{\varphi}_{\gamma} D^{m,\delta}_{y}}{\delta^{m-\beta}\Gamma(m-\beta)} \int_y^z \exp\left[\frac{\delta-1}{\delta}(\varphi(u)-\varphi(y))\right] \\ &\cdot (\varphi(u)-\varphi(y))^{m-\beta-1}\varphi'(u)g(u)\mathrm{d}u,\end{aligned} \tag{22}$$

where

$$\varphi D^{m,\delta} = \underbrace{\varphi D^{\delta\varphi} D^{\delta} \ldots . . \varphi D^{\delta}}_{m-\text{times}}, m = [\mathrm{Re}(\beta)] + 1, \tag{23}$$

$$\begin{aligned}\left({}^{\varphi}_{\gamma} D^{\delta} g\right)(y) &= (1-\delta)g(y) - \delta\frac{g'(y)}{\varphi'(y)}, {}^{\varphi}_{\gamma} D^{m,\delta} \\ &= \underbrace{{}^{\varphi}_{\gamma} D^{\delta\varphi}_{\gamma} D^{\delta} \ldots . . {}^{\varphi}_{\gamma} D^{\delta}}_{m-\text{times}}.\end{aligned} \tag{24}$$

The notations $(\varphi D^{\beta,\delta}_{a^+} g)(y)$ and $(\varphi D^{\beta,\delta}_{z^-} g)(y)$ are, respectively, called the left- and right-sided proportional fractional derivatives of a function g with respect to φ for the order β.

Definition 7 (see [30]). For the integrable function g and for the continuous and strictly increasing function φ on $[a,z]$, let $\delta \in (0,1]$, and we have for all $\beta \in \mathbb{C}$ and $\mathrm{Re}(\beta) \geq 0$,

$$\begin{aligned}\left(\varphi \mathcal{I}^{\beta,\delta}_{a^+} g\right)(y) &= \frac{1}{\delta^{\beta}\Gamma(\beta)} \int_a^y \exp\left[\frac{\delta-1}{\delta}(\varphi(y)-\varphi(u))\right] \\ &\cdot (\varphi(y)-\varphi(u))^{\beta-1}\varphi'(u)g(u)\mathrm{d}u,\end{aligned} \tag{25}$$

$$\begin{aligned}\left(\varphi \mathcal{I}^{\beta,\delta}_{z^-} g\right)(y) &= \frac{1}{\delta^{\beta}\Gamma(\beta)} \int_y^z \exp\left[\frac{\delta-1}{\delta}(\varphi(u)-\varphi(y))\right] \\ &\cdot (\varphi(u)-\varphi(y))^{\beta-1}\varphi'(u)g(u)\mathrm{d}u.\end{aligned} \tag{26}$$

The notations $(\varphi \mathcal{I}^{\beta,\delta}_{a^+} g)(y)$ and $(\varphi \mathcal{I}^{\beta,\delta}_{z^-} g)(y)$ are, respectively, called the left- and right-sided proportional fractional integrals of a function g with respect to φ for the order β.

Lemma 2 (see [30]). *Let φ be a continuous function on $y \geq a$. If $\delta \in (0,1]$ and $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, we have*

$$\begin{aligned}\varphi \mathcal{I}^{\beta,\delta}_{a^+}\left(\varphi \mathcal{I}^{\alpha,\delta}_{a^+} g\right)(y) &= \varphi \mathcal{I}^{\alpha,\delta}_{a^+}\left(\varphi \mathcal{I}^{\beta,\delta}_{a^+} g\right)(y) \\ &= \left(\varphi \mathcal{I}^{\beta+\alpha,\delta}_{a^+} g\right)(y),\end{aligned} \tag{27}$$

$$\begin{aligned}\varphi \mathcal{I}^{\beta,\delta}_{z^-}\left(\varphi \mathcal{I}^{\alpha,\delta}_{z^-} g\right)(y) &= \varphi \mathcal{I}^{\beta,v}_{z^-}\left(\varphi \mathcal{I}^{\alpha,\delta}_{z^-} g\right)(y) \\ &= \left(\varphi \mathcal{I}^{\beta+\alpha,\delta}_{z^-} g\right)(y).\end{aligned} \tag{28}$$

Lemma 3 (see [30]). *Let φ be an integrable function defined on $[a,y]$ and $\delta > a$. If $0 \leq m < [\mathrm{Re}(\beta)] + 1$, then we have*

$$\varphi D^{m,\delta}\left(\varphi \mathcal{I}^{\beta,\delta}_{a^+} g\right)(y) = \left(\varphi \mathcal{I}^{\beta-m,\delta}_{a^+} g\right)(y), \tag{29}$$

$${}^{\varphi}_{\gamma} D^{m,\delta}\left(\varphi \mathcal{I}^{\beta,\delta}_{z^-} g\right)(y) = \left(\varphi \mathcal{I}^{\beta-m,\delta}_{z^-} g\right)(y). \tag{30}$$

In this paper, we need the following identity as in [31].

Let $\delta \in (0,1]$, $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) \geq 0$, and φ be a strictly increasing continuous function. Then, for any constant k, we have

$$\left(\varphi \mathcal{I}^{\beta,\delta}_{x^+} k\right)(y) = \frac{(\varphi(y)-\varphi(x))^{\beta}}{\delta^{\beta}\Gamma(\beta+1)} k. \tag{31}$$

3. Fractional Hermite–Hadamard–Mercer Inequalities Involving φ-Proportional Fractional Integrals

This section is the first part of our main contributions. Here, we present basic generalization in Hermite–Hadamard–Mercer's inequalities which involve convex functions for generalized proportional fractional integral operators concerning another strictly increasing continuous function.

Theorem 6. *Let* $\varphi\colon \mathrm{F} \longrightarrow [a,z]\subseteq\mathbb{R}$, *with* $0\leq a<z$, *be a continuous strictly increasing function and* $g\colon [a,z]\longrightarrow\mathbb{R}$ *be a convex differentiable function on* (a,z) *satisfying that* $(g^{\circ}\varphi)\colon \mathrm{F}\longrightarrow\mathbb{R}$ *is an integrable mapping on* F. *Then, we have*

$$\begin{aligned} g\left(a+z-\frac{x+y}{2}\right) &\leq g(a)+g(z)-\frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(g^{\circ}\varphi)\left(\varphi^{-1}(y)\right)+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(g^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\} \\ &\leq g(a)+g(z)-g\left(\frac{x+y}{2}\right), \end{aligned} \quad (32)$$

$$\begin{aligned} g\left(a+z-\frac{x+y}{2}\right) &\leq \frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\right. \\ &\quad \left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right\} \\ &\leq \frac{g(a+z-y)+g(a+z-x)}{2}\leq g(a)+g(z)-\frac{g(x)+g(y)}{2}. \end{aligned} \quad (33)$$

Proof. According to the Jensen–Mercer inequality and for $r,s\in[a,z]$, we have

$$g\left(a+z-\frac{r+s}{2}\right)\leq g(a)+g(z)-\frac{g(r)+g(s)}{2}. \quad (34)$$

Now, we change the variables r and s with $r=\eta x+(1-\eta)y$ and $s=(1-\eta)x+\eta y$, and we get

$$\begin{aligned} g&\left(a+z-\frac{\eta x+(1-\eta)y+(1-\eta)x+\eta y}{2}\right) \\ &\leq g(a)+g(z)-\frac{g(\eta x+(1-\eta)y)+g((1-\eta)x+\eta y)}{2}, \end{aligned} \quad (35)$$

which leads to

$$\begin{aligned} g\left(a+z-\frac{x+y}{2}\right)&\leq g(a)+g(z) \\ &\quad -\frac{g(\eta x+(1-\eta)y)+g((1-\eta)x+\eta y)}{2}. \end{aligned} \quad (36)$$

On both sides of (36), taking product by $\exp[\delta-1/\delta\eta(y-x)]\eta^{\beta-1}$ and then integrating the estimating inequality with respect to η over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 g\left(a+z-\frac{x+y}{2}\right)\exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}d\eta \\ &\leq \int_0^1[g(a)+g(z)]\exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}d\eta \\ &\quad -\int_0^1\exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\frac{g(\eta x+(1-\eta)y)+g((1-\eta)x+\eta y)}{2}\eta^{\beta-1}d\eta. \end{aligned} \quad (37)$$

Using identity (31) on both sides of (37), we obtain

$$\frac{2}{\beta}g\left(a+z-\frac{x+y}{2}\right)$$
$$\leq\frac{2}{\beta}\left[g(a)+g(z)\right]-\int_0^1\exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]$$
$$\cdot\eta^{\beta-1}g(\eta x+(1-\eta)y)d\eta$$
$$+\int_0^1\exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}g((1-\eta)x+\eta y)d\eta. \tag{38}$$

Next,

$$\begin{aligned}&\frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(g^{\circ}\varphi)\left(\varphi^{-1}(y)\right)+{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(g^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\}\\&=\frac{\beta}{2(y-x)^{\beta}}\left\{\int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)}\exp\left[\frac{\delta-1}{\delta}(y-\varphi(u))\right](y-\varphi(u))^{\beta-1}(g^{\circ}\varphi)(u)\varphi'(u)\mathrm{d}u\right.\\&\left.+\int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)}\exp\left[\frac{\delta-1}{\delta}(\varphi(v)-x)\right](\varphi(v)-x)^{\beta-1}(g^{\circ}\varphi)(v)\varphi'(v)\mathrm{d}v\right\}\\&=\frac{\beta}{2}\left\{\int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)}\exp\left[\frac{\delta-1}{\delta}(y-\varphi(u))\right]\left(\frac{y-\varphi(u)}{y-x}\right)^{\beta-1}(g^{\circ}\varphi)(u)\frac{\varphi'(u)}{y-x}\mathrm{d}u\right.\\&\left.+\int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)}\exp\left[\frac{\delta-1}{\delta}(\varphi(v)-x)\right]\left(\frac{\varphi(v)-x}{y-x}\right)^{\beta-1}(g^{\circ}\varphi)(v)\frac{\varphi'(v)}{y-x}\mathrm{d}v\right\}.\end{aligned} \tag{39}$$

Putting $\varphi(u)=\eta x+(1-\eta)y$ and $\varphi(v)=(1-\eta)x+\eta y$, we get

$$\begin{aligned}&g(a)+g(z)-\frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(g^{\circ}\varphi)\left(\varphi^{-1}(y)\right)+{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(g^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\}\\&=g(a)+g(z)-\frac{\beta}{2}\left\{\int_0^1\exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}g(\eta x+(1-\eta)y)d\eta\right.\\&\left.+\int_0^1\exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}g((1-\eta)x+\eta y)d\eta\right\}\\&\geq g\left(a+z-\frac{x+y}{2}\right).\end{aligned} \tag{40}$$

This proves the first inequality in (32). To prove the second inequality and by using the convexity of g, we can be certain that

$$g\left(\frac{\varphi(u)+\varphi(v)}{2}\right)\leq\frac{(g^{\circ}\varphi)(u)+(g^{\circ}\varphi)(v)}{2}. \tag{41}$$

Then, for $\varphi(u)=\eta x+(1-\eta)y$ and $\varphi(v)=(1-\eta)x+\eta y$, we have

$$g\left(\frac{x+y}{2}\right)\leq\frac{g(\eta x+(1-\eta)y)+g((1-\eta)x+\eta y)}{2},\quad\eta\in[0,1]. \tag{42}$$

On both sides of (42), taking product by $\exp[\delta-1/\delta\eta(y-x)]\eta^{\beta-1}$ and then integrating the estimating inequality with respect to η over $[0,1]$, we get

$$g\left(\frac{x+y}{2}\right) \leq \frac{\beta}{2}\left\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1} g(\eta x+(1-\eta)y)d\eta + \int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1} g((1-\eta)x+\eta y)d\eta\right\} = \frac{\delta^\beta\Gamma(\beta+1)}{2(y-x)^\beta}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^+}(g^\circ\varphi)(\varphi^{-1}(y)) + \varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^-}(g^\circ\varphi)(\varphi^{-1}(x))\right\}. \tag{43}$$

Therefore, we have

$$-g\left(\frac{x+y}{2}\right) \geq -\frac{\delta^\beta\Gamma(\beta+1)}{2(y-x)^\beta}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^+}(g^\circ\varphi)(\varphi^{-1}(y)) + \varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^-}(g^\circ\varphi)(\varphi^{-1}(x))\right\}. \tag{44}$$

On both sides of inequality (44), adding $g(a)+g(z)$, we get the second inequality in (32). Hence, desired inequality (32) is thus proved. We now give the proof of inequalities in (33). We have, according to the convexity of the function g, for all $r,s\in[a,z]$, that

$$g\left(a+z-\frac{r+s}{2}\right) = g\left(\frac{a+z-r+a+z-s}{2}\right) \leq \frac{1}{2}\{g(a+z-r)+g(a+z-s)\}. \tag{45}$$

By applying the change of variables $a+z-r=\eta(a+z-x)+(1-\eta)(a+z-y)$ and $a+z-s=\eta(a+z-y)+(1-\eta)(a+z-x)$, we obtain

$$g\left(a+z-\frac{x+y}{2}\right) \leq \frac{1}{2}\{g(\eta(a+z-x)+(1-\eta)(a+z-y)) + g(\eta(a+z-y)+(1-\eta)(a+z-x))\}. \tag{46}$$

On both sides of (46), taking product by $\exp[\delta-1/\delta\eta(y-x)]\eta^{\beta-1}$ and then integrating the estimating inequality with respect to η over $[0,1]$, we obtain

$$g\left(a+z-\frac{x+y}{2}\right) \leq \frac{\beta}{2}\left\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1} g[\eta(a+z-x)+(1-\eta)(a+z-y)]d\eta + \int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1} g[\eta(a+z-y)+(1-\eta)(a+z-x)]d\eta\right\}. \tag{47}$$

Next,

$$\begin{aligned}
&\frac{\delta^\beta\Gamma(\beta+1)}{2(y-x)^\beta}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(g^\circ\varphi)(\varphi^{-1}(a+z-x)) + \varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(g^\circ\varphi)(\varphi^{-1}(a+z-y))\right\}\\
&= \frac{\beta}{2(y-x)^\beta}\left\{\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)} \exp\left[\frac{\delta-1}{\delta}(\{a+z-x\}-\varphi(u))\right](\{a+z-x\}-\varphi(u))^{\beta-1}(g^\circ\varphi)(u)\varphi'(u)\mathrm{d}u\right.\\
&\quad\left. + \int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)} \exp\left[\frac{\delta-1}{\delta}(\varphi(v)-\{a+z-y\})\right](\varphi(v)-\{a+z-y\})^{\beta-1}(g^\circ\varphi)(v)\varphi'(v)\mathrm{d}v\right\}\\
&= \frac{\beta}{2}\left\{\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)} \exp\left[\frac{\delta-1}{\delta}(\{a+z-x\}-\varphi(u))\right]\left(\frac{\{a+z-x\}-\varphi(u)}{y-x}\right)^{\beta-1}(g^\circ\varphi)(u)\frac{\varphi'(u)}{y-x}\mathrm{d}u\right.\\
&\quad\left. + \int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)} \exp\left[\frac{\delta-1}{\delta}(\varphi(v)-\{a+z-y\})\right]\left(\frac{\varphi(v)-\{a+z-y\}}{y-x}\right)^{\beta-1}(g^\circ\varphi)(v)\frac{\varphi'(v)}{y-x}\mathrm{d}v\right\}.
\end{aligned} \tag{48}$$

Putting $\varphi(u) = \eta(a+z-y) + (1-\eta)(a+z-x)$ and $\varphi(v) = \eta(a+z-x) + (1-\eta)(a+z-y)$, we get

$$\begin{aligned}
&\frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right) + \varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right\} \\
&= \frac{\beta}{2}\left\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}g[\eta(a+z-x)+(1-\eta)(a+z-y)]d\eta\right. \\
&\left.+\int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}g[\eta(a+z-y)+(1-\eta)(a+z-x)]d\eta\right\} \\
&\geq g\left(a+z-\frac{x+y}{2}\right).
\end{aligned} \tag{49}$$

This completes the proof of the first inequality in (33). To prove the second inequality and by using the convexity of g, we can be certain that

$$g[\eta(a+z-x)+(1-\eta)(a+z-y)] \leq \eta g(a+z-x) + (1-\eta)g(a+z-y), \tag{50}$$

$$g[\eta(a+z-y)+(1-\eta)(a+z-x)] \leq \eta g(a+z-y) + (1-\eta)g(a+z-x). \tag{51}$$

Adding inequalities (50) and (51), we obtain

$$\begin{aligned}
&g[\eta(a+z-x)+(1-\eta)(a+z-y)] + g[\eta(a+z-y)+(1-\eta)(a+z-x)] \\
&\quad\leq g(a+z-y) + g(a+z-x) \leq 2[g(a)+g(z)] - [g(x)+g(y)].
\end{aligned} \tag{52}$$

On both sides of (52), taking product by $\exp[\delta-1/\delta\eta(y-x)]\eta^{\beta-1}$ and then integrating the estimating inequality with respect to η over $[0,1]$, we obtain

$$\begin{aligned}
&\left\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}g[\eta(a+z-x)+(1-\eta)(a+z-y)]d\eta\right. \\
&\left.\quad+\int_0^1 \exp\left[\frac{\delta-1}{\delta}\eta(y-x)\right]\eta^{\beta-1}g[\eta(a+z-y)+(1-\eta)(a+z-x)]d\eta\right\} \\
&\leq \frac{1}{\beta}g(a+z-y) + \frac{1}{\beta}g(a+z-x) \\
&\leq \frac{2}{\beta}[g(a)+g(z)] - \frac{1}{\beta}[g(x)+g(y)].
\end{aligned} \tag{53}$$

On the left-hand side in (53), applying the same arguments as above, we obtain

$$\begin{aligned}&\frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)+{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right\}\\&\leq\frac{g(a+z-y)+g(a+z-x)}{2}\leq g(a)+g(z)-\frac{g(x)+g(y)}{2},\end{aligned}\tag{54}$$

which is the second and third inequalities in (33). Hence, the desired inequalities in (33) are thus proved. □

Remark 1

The proportional fractional integral version of Theorem 6 was provided by K. Yildirim and S. Yildirim in [32].

If we put $\delta = 1$, in Theorem 6, we obtain its φ-Riemann–Liouville fractional integral version, which was proved by Butt et al. in [33].

Fixing $\delta = 1$ and $\varphi(x) = x$ in Theorem 6 for all $x \in [a, z]$, it gives

$$\begin{aligned}g\left(a+z-\frac{x+y}{2}\right)&\leq g(a)+g(z)-\frac{\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\mathcal{I}^{\beta}_{x^{+}}g(y)+\mathcal{I}^{\beta}_{y^{-}}g(x)\right\}\\&\leq g(a)+g(z)-g\left(\frac{x+y}{2}\right),\end{aligned}\tag{55}$$

$$\begin{aligned}g\left(a+z-\frac{x+y}{2}\right)&\leq\frac{\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{\mathcal{I}^{\beta}_{(a+z-y)^{+}}g(a+z-x)+\mathcal{I}^{\beta}_{(a+z-x)^{-}}g(a+z-y)\right\}\\&\leq\frac{g(a+z-y)+g(a+z-x)}{2}\leq g(a)+g(z)-\frac{g(x)+g(y)}{2}.\end{aligned}\tag{56}$$

This was given by Kian and Moslehian in [34].

This was proved by Ogulmus and Sarikaya in [19].

Fixing $\delta = 1$, $\beta = 1$, and $\varphi(x) = x$ in Theorem 6 for all $x \in [a, z]$, it gives

$$\begin{aligned}g\left(a+z-\frac{x+y}{2}\right)&\leq g(a)+g(z)-\int_{0}^{1}g(\eta x+(1-\eta)y)d\eta\\&\leq g(a)+g(z)-g\left(\frac{x+y}{2}\right),\end{aligned}\tag{57}$$

$$\begin{aligned}g\left(a+z-\frac{x+y}{2}\right)&\leq\frac{1}{y-x}\int_{x}^{y}g(a+z-\eta)d\eta\\&\leq g(a)+g(z)-\frac{g(x)+g(y)}{2}.\end{aligned}\tag{58}$$

Theorem 7. *Let φ: $\digamma \longrightarrow [a, z] \subseteq \mathbb{R}$, with $0 \leq a < z$, be a continuous strictly increasing function and g: $[a, z] \longrightarrow \mathbb{R}$ be a convex differentiable function on (a, z) satisfying that $(g^{\circ}\varphi)$: $\digamma \longrightarrow \mathbb{R}$ is an integrable mapping on $\digamma$. Then, we have*

$$\begin{aligned}&g\left(a+z-\frac{x+y}{2}\right)\\&\leq\frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\left\{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)\right\}^{-}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right.\\&\left.\quad+{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\left\{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)\right\}^{+}}(g^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\}\\&\leq g(a)+g(z)-\frac{g(x)+g(y)}{2}.\end{aligned}\tag{59}$$

Proof. According to the convexity of the function g for all $r, s \in [a, z]$, we have

$$g\left(a+z-\frac{r+s}{2}\right) = g\left(\frac{a+z-r+a+z-s}{2}\right) \leq \frac{1}{2}\{g(a+z-r)+g(a+z-s)\}. \quad (60)$$

Putting $r = \eta/2x + 2 - \eta/2y$ and $s = 2 - \eta/2x + \eta/2y$, it follows, for all $r, s \in [a, z]$ and $\eta \in [0, 1]$, that

$$g\left(a+z-\frac{x+y}{2}\right) \leq \frac{1}{2}\left\{g\left(a+z-\left(\frac{\eta}{2}x+\frac{2-\eta}{2}y\right)\right) + g\left(a+z-\left(\frac{2-\eta}{2}x+\frac{\eta}{2}y\right)\right)\right\}. \quad (61)$$

On both sides of (61), taking product by $\exp[\delta-1/\delta\eta/2(y-x)](\eta/2)^{\beta-1}$ and then integrating the estimating inequality with respect to η over $[0, 1]$, we obtain

$$\begin{aligned}\frac{1}{\beta}g\left(a+z-\frac{x+y}{2}\right) &\leq \int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\eta}{2}(y-x)\right]\left(\frac{\eta}{2}\right)^{\beta-1} \cdot g\left(a+z-\left(\frac{\eta}{2}x+\frac{2-\eta}{2}y\right)\right)d\eta \\ &+ \int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\eta}{2}(y-x)\right]\left(\frac{\eta}{2}\right)^{\beta-1} \cdot g\left(a+z-\left(\frac{2-\eta}{2}x+\frac{\eta}{2}y\right)\right)d\eta.\end{aligned} \quad (62)$$

Next,

$$\begin{aligned}&\frac{\delta^\beta\Gamma(\beta+1)}{2(y-x)^\beta}\left\{{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\left\{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)\right\}^-}(g^\circ\varphi)\left(\varphi^{-1}(a+z-y)\right) + {}_{\varphi}\mathcal{I}^{\beta,\delta}_{\left\{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)\right\}^+}(g^\circ\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\} \\ &= \frac{\beta}{2(y-x)^\beta}\left\{\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)} \exp\left[\frac{\delta-1}{\delta}(\varphi(u)-\{a+z-y\})\right](\varphi(u)-\{a+z-y\})^{\beta-1}(g^\circ\varphi)(u)\varphi'(u)du\right. \\ &\left.+ \int_{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)}^{\varphi^{-1}(a+z-x)} \exp\left[\frac{\delta-1}{\delta}(\{a+z-x\}-\varphi(v))\right](\{a+z-x\}-\varphi(v))^{\beta-1}(g^\circ\varphi)(v)\varphi'(v)dv\right\} \\ &= \beta\left\{\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)} \exp\left[\frac{\delta-1}{\delta}(\varphi(u)-\{a+z-y\})\right]\left(\frac{\varphi(u)-\{a+z-y\}}{y-x}\right)^{\beta-1}(g^\circ\varphi)(u)\frac{\varphi'(u)}{y-x}du\right. \\ &\left.+ \int_{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)}^{\varphi^{-1}(a+z-x)} \exp\left[\frac{\delta-1}{\delta}(\{a+z-x\}-\varphi(v))\right]\left(\frac{\{a+z-x\}-\varphi(v)}{y-x}\right)^{\beta-1}(g^\circ\varphi)(v)\frac{\varphi'(v)}{y-x}dv\right\}.\end{aligned} \quad (63)$$

Putting $\varphi(u) = a + z - (\eta/2x + 2 - \eta/2y)$ and $\varphi(v) = a + z - (2 - \eta/2x + \eta/2y)$, we obtain

$$\begin{aligned}&\frac{\delta^\beta\Gamma(\beta+1)}{2(y-x)^\beta}\left\{{}_{\varphi}\mathcal{I}^{\beta,\delta}_{\left\{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)\right\}^-}(g^\circ\varphi)\left(\varphi^{-1}(a+z-y)\right) + {}_{\varphi}\mathcal{I}^{\beta,\delta}_{\left\{\varphi^{-1}\left(a+z-\frac{x+y}{2}\right)\right\}^+}(g^\circ\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\} \\ &= \beta\left\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\eta}{2}(y-x)\right]\left(\frac{\eta}{2}\right)^{\beta-1} g\left(a+z-\left(\frac{\eta}{2}x+\frac{2-\eta}{2}y\right)d\eta\right)\right. \\ &\left.+ \int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\eta}{2}(y-x)\right]\left(\frac{\eta}{2}\right)^{\beta-1} g\left(a+z-\left(\frac{2-\eta}{2}x+\frac{\eta}{2}y\right)\right)d\eta\right\} \\ &\geq g\left(a+z-\frac{x+y}{2}\right).\end{aligned} \quad (64)$$

This proves the first inequality in (59). To prove the second inequality and by using the Jensen–Mercer inequality, we can be certain that

$$g\left(a+z-\left(\frac{\eta}{2}x+\frac{2-\eta}{2}y\right)\right)\leq g(a)+g(z)-\left(\frac{\eta}{2}g(x)+\frac{2-\eta}{2}g(y)\right), \tag{65}$$

$$g\left(a+z-\left(\frac{2-\eta}{2}x+\frac{\eta}{2}y\right)\right)\leq g(a)+g(z)-\left(\frac{2-\eta}{2}g(x)+\frac{\eta}{2}g(y)\right). \tag{66}$$

Adding inequalities (65) and (66), we get

$$\begin{aligned} &g\left(a+z-\left(\frac{\eta}{2}x+\frac{2-\eta}{2}y\right)\right)+g\left(a+z-\left(\frac{2-\eta}{2}x+\frac{\eta}{2}y\right)\right)\\ &\leq 2[g(a)+g(z)]-[g(x)+g(y)]. \end{aligned} \tag{67}$$

On both sides of (67), taking product by $\exp[\delta-1/\delta\eta/2(y-x)](\eta/2)^{\beta-1}$ and then integrating the estimating inequality with respect to η over $[0,1]$, we obtain

$$\begin{aligned} &\beta\Bigg\{\int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\eta}{2}(y-x)\right]\left(\frac{\eta}{2}\right)^{\beta-1} g\left(a+z-\left(\frac{\eta}{2}x+\frac{2-\eta}{2}y\right)\right)d\eta\\ &\quad+\int_0^1 \exp\left[\frac{\delta-1}{\delta}\frac{\eta}{2}(y-x)\right]\left(\frac{\eta}{2}\right)^{\beta-1} g\left(a+z-\left(\frac{2-\eta}{2}x+\frac{\eta}{2}y\right)\right)d\eta\Bigg\}\\ &\leq g(a)+g(z)-\frac{1}{2}[g(x)+g(y)]. \end{aligned} \tag{68}$$

By comparing the left-hand side of inequality (64) with the left-hand side of inequality (68), we can deduce

$$\begin{aligned} &\frac{\delta^{\beta}\Gamma(\beta+1)}{2(y-x)^{\beta}}\left\{{}^{\varphi}\mathcal{J}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x+y/2)\}^{-}}\left(g^{\circ}\varphi\right)\left(\varphi^{-1}(a+z-y)\right)+{}^{\varphi}\mathcal{J}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x+y/2)\}^{+}}\left(g^{\circ}\varphi\right)\left(\varphi^{-1}(a+z-x)\right)\right\}\\ &\leq g(a)+g(z)-\frac{g(x)+g(y)}{2}, \end{aligned} \tag{69}$$

which is the second inequality in (59). The proof is thus completed. □

4. Weighted Fractional Hermite–Hadamard–Mercer Inequalities Involving φ-Proportional Fractional Integrals

This section is the second part of our main contributions, within which we give the fractional weighted Hermite–Hadamar–Mercer's inequalities which involve convex functions for generalized proportional fractional integral operators concerning another strictly increasing continuous function.

Theorem 8. *Let $\varphi\colon \mathsf{F}\longrightarrow[a,z]\subseteq\mathbb{R}$, with $0\leq a<z$, be a continuous strictly increasing function, $g\colon[a,z]\longrightarrow\mathbb{R}$ be a convex differentiable function on (a,z), and $w\colon[a,z]\longrightarrow\mathbb{R}$ be a nonnegative integrable function satisfying that $(g^{\circ}\varphi)$, $(w^{\circ}\varphi)\colon\mathsf{F}\longrightarrow\mathbb{R}$ are integrable mappings on F. Then, we have*

$$\begin{aligned}
&g\left(a+z-\frac{x+y}{2}\right)^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}\left(w^{\circ}\varphi\right)\left(\varphi^{-1}(a+z-y)\right)\\
&\leq\frac{1}{2}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}\left(g^{\circ}\varphi\right)\left(w^{\circ}\varphi\right)\left(\varphi^{-1}(a+z-y)\right)\right.\\
&\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}\left(g^{\circ}\varphi\right)\left(w^{\circ}\varphi\right)\left(\varphi^{-1}(a+z-x)\right)\right\}\\
&\leq\frac{1}{2}[g(a+z-x)+g(a+z-y)]^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}\left(w^{\circ}\varphi\right)\left(\varphi^{-1}(a+z-y)\right)\\
&\leq\left\{g(a)+g(z)-\frac{g(x)+g(y)}{2}\right\}^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}\left(w^{\circ}\varphi\right)\left(\varphi^{-1}(a+z-y)\right).
\end{aligned} \tag{70}$$

Proof. According to the convexity of the function g on $[a,z]$, we have

$$\begin{aligned}
g\left(a+z-\frac{x+y}{2}\right)&=g\left(\frac{1}{2}(a+z-(\eta x+(1-\eta)y)+a+z-(\eta y+(1-\eta)x))\right)\\
&\leq\frac{g(a+z-(\eta x+(1-\eta)y))+g(a+z-(\eta y+(1-\eta)x))}{2}.
\end{aligned} \tag{71}$$

On both sides of (71), taking product by $h(a+z-(\eta x+(1-\eta)y))$ and then integrating the estimating inequality with respect to η over $[0,1]$, we obtain

$$\begin{aligned}
&g\left(a+z-\frac{x+y}{2}\right)\int_0^1 h(a+z-(\eta x+(1-\eta)y))d\eta\\
&\leq\frac{1}{2}\left\{\int_0^1 g(a+z-(\eta x+(1-\eta)y))h(a+z-(\eta x+(1-\eta)y))d\eta\right.\\
&\left.+\int_0^1 g(a+z-(\eta y+(1-\eta)x))h(a+z-(\eta x+(1-\eta)y))d\eta\right\}.
\end{aligned} \tag{72}$$

Applying the change of variables $\varphi(u)=a+z-(\eta x+(1-\eta)y)$, to each of the left-hand side and the first integration in (72) and applying the changeof variables $\varphi(u)=a+z-(\eta y+(1-\eta)x)$, to the second integration on the right-hand side, we get

$$\begin{aligned}
&g\left(a+z-\frac{x+y}{2}\right)\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)}\left(h^{\circ}\varphi\right)(u)\frac{\varphi'(u)}{y-x}\mathrm{d}u\\
&\leq\frac{1}{2}\left\{\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)}\left(g^{\circ}\varphi\right)(u)\left(h^{\circ}\varphi\right)(u)\frac{\varphi'(u)}{y-x}\mathrm{d}u\right.\\
&\left.+\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)}\left(g^{\circ}\varphi\right)(u)h(2(a+z)-(x+y+\varphi(u)))\frac{\varphi'(u)}{y-x}\mathrm{d}u\right\}.
\end{aligned} \tag{73}$$

Choosing

$$(h^{\circ}\varphi)(u) = \frac{1}{\delta^{\beta}\Gamma(\beta)}\exp\left[\frac{\delta-1}{\delta}(\varphi(u)-\{a+z-y\})\right] \cdot (\varphi(u)-\{a+z-y\})^{\beta-1}(w^{\circ}\varphi)(u), \tag{74}$$

then (73) leads to

$$\begin{aligned}&\frac{1}{y-x}g\left(a+z-\frac{x+y}{2}\right)^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\\&\leq\frac{1}{2(y-x)}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right.\\&\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\},\end{aligned} \tag{75}$$

which proves the first inequality in (70). To prove the second inequality, by using the convexity of g, we have

$$\begin{aligned}g(a+z-(\eta x+(1-\eta)y)) &= g(\eta(a+z-x)+(1-\eta)(a+z-y))\\&\leq\eta g(a+z-x)+(1-\eta)g(a+z-y),\end{aligned} \tag{76}$$

$$\begin{aligned}g(a+z-(\eta y+(1-\eta)x)) &= g(\eta(a+z-y)+(1-\eta)(a+z-x))\\&\leq\eta g(a+z-y)+(1-\eta)g(a+z-x).\end{aligned} \tag{77}$$

Adding inequalities (76) and (77), we obtain

$$\begin{aligned}&g(a+z-(\eta x+(1-\eta)y))+g(a+z-(\eta y+(1-\eta)x))\\&\quad\leq g(a+z-x)+g(a+z-y).\end{aligned} \tag{78}$$

On both sides of (78), taking product by $1/2h(a+z-(\eta x+(1-\eta)y))$ and then integrating the estimating inequality with respect to η over $[0,1]$, we obtain

$$\begin{aligned}&\frac{1}{2}\left\{\int_0^1 g(a+z-(\eta x+(1-\eta)y))h(a+z-(\eta x+(1-\eta)y))d\eta\right.\\&\left.+\int_0^1 g(a+z-(\eta y+(1-\eta)x))h(a+z-(\eta x+(1-\eta)y))d\eta\right\}\\&\leq\frac{1}{2}\int_0^1[g(a+z-x)+g(a+z-y)]h(a+z-(\eta x+(1-\eta)y))d\eta.\end{aligned} \tag{79}$$

Applying the change of variables $\varphi(u)=a+z-(\eta y+(1-\eta)x)$, to the second integration in the left-hand side of (79) and applying the change of variables $\varphi(u)=a+z-(\eta x+(1-\eta)y)$, to the first integration on left-hand side and to the right-hand side, then we obtain

$$\begin{aligned}&\frac{1}{2(y-x)}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(g^\circ\varphi)(w^\circ\varphi)\left(\varphi^{-1}(a+z-y)\right)\right.\\&\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(g^\circ\varphi)(w^\circ\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\}\\&\leq\frac{1}{2(y-x)}[g(a+z-x)+g(a+z-y)]^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(w^\circ\varphi)\left(\varphi^{-1}(a+z-y)\right),\end{aligned}\tag{80}$$

and the second inequality is thus proved. To prove the third inequality in (70), by using the convexity of g and Lemma 1, we have

$$g(a+z-x)\leq g(a)+g(z)-g(x),\tag{81}$$

$$g(a+z-y)\leq g(a)+g(z)-g(y).\tag{82}$$

Adding inequalities (81) and (82), we get

$$g(a+z-x)+g(a+z-y)\leq 2[g(a)+g(z)]-g(x)-g(y).\tag{83}$$

On both sides of (83), taking product by $1/2h(a+z-(\eta x+(1-\eta)y))$, integrating the estimating inequality with respect to η over $[0,1]$, and then applying the changing of variables $\varphi(u)=a+z-(\eta x+(1-\eta)y)$, we get

$$\begin{aligned}&\frac{g(a+z-x)+g(a+z-y)}{2}\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)}(h^\circ\varphi)(u)\frac{\varphi'(u)}{y-x}\mathrm{d}u\\&\leq\left\{g(a)+g(z)-\frac{g(x)+g(y)}{2}\right\}\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)}(h^\circ\varphi)(u)\frac{\varphi'(u)}{y-x}\mathrm{d}u.\end{aligned}\tag{84}$$

By choosing

$$\begin{aligned}(h^\circ\varphi)(u)&=\frac{1}{\delta^{\beta}\Gamma(\beta)}\exp\left[\frac{\delta-1}{\delta}(\varphi(u)-\{a+z-y\})\right]\\&\quad\cdot(\varphi(u)-\{a+z-y\})^{\beta-1}(w^\circ\varphi)(u),\end{aligned}\tag{85}$$

we can reach the third desired inequality in (70). Hence, the proof is thus completed. □ □

Corollary 1. *Let $\varphi\colon \digamma\longrightarrow[a,z]\subseteq\mathbb{R}$, with $0\leq a<z$, be a continuous strictly increasing function, $g\colon[a,z]\longrightarrow\mathbb{R}$ be a convex differentiable function on (a,z), and $w\colon[a,z]\longrightarrow\mathbb{R}$ be a nonnegative integrable function satisfying that $(g^\circ\varphi)$, $(w^\circ\varphi)\colon\digamma\longrightarrow\mathbb{R}$ are integrable mappings on $\digamma$. Then, we have*

$$\begin{aligned}&g\left(a+z-\frac{x+y}{2}\right)^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(w^\circ\varphi)\left(\varphi^{-1}(a+z-x)\right)\\&\leq\frac{1}{2}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(g^\circ\varphi)(w^\circ\varphi)\left(\varphi^{-1}(a+z-x)\right)\right.\\&\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(g^\circ\varphi)(w^\circ\varphi)\left(\varphi^{-1}(a+z-y)\right)\right\}\\&\leq\frac{1}{2}[g(a+z-x)+g(a+z-y)]^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(w^\circ\varphi)\left(\varphi^{-1}(a+z-x)\right)\\&\leq\left\{g(a)+g(z)-\frac{g(x)+g(y)}{2}\right\}^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(w^\circ\varphi)\left(\varphi^{-1}(a+z-x)\right).\end{aligned}\tag{86}$$

Proof. This corollary can be easily demonstrated by following the proof of Theorem 8, taking

$$(h^\circ\varphi)(u)=\frac{1}{\delta^{\beta}\Gamma(\beta)}\exp\left[\frac{\delta-1}{\delta}(\{a+z-x\}-\varphi(u))\right](\{a+z-x\}-\varphi(u))^{\beta-1}(w^\circ\varphi)(u).\tag{87}$$

□

Remark 2. By adding inequalities (70) and (86), we can derive the following inequality:

$$\begin{aligned}
&g\left(a+z-\frac{x+y}{2}\right)\Big\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\\
&\quad+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\Big\}\\
&\le\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\\
&\quad+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\\
&\le\frac{1}{2}\left[g(a+z-x)+g(a+z-y)\right]\Big\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\\
&\quad+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\Big\}\\
&\le\left\{g(a)+g(z)-\frac{g(x)+g(y)}{2}\right\}\Big\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\\
&\quad+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\Big\}.
\end{aligned}\tag{88}$$

Remark 3

- Putting $\delta=1$, we obtain the proportional fractional integral version of each of Theorem 8, Corollary 1, and inequality (88) with respect to the positive increasing function φ
- By taking $\varphi(x)=x$, for all $x\in[a,z]$, we obtain the proportional fractional integral version for each of Theorem 8, Corollary 1, and inequality (88)
- If we put $\delta=1$ and $\varphi(x)=x$, for all $x\in[a,z]$, we obtain inequality (9), for Riemann–Liouville fractional integral introduced by Iscan [20]
- Putting $\delta=1$ and $\varphi(x)=x$, for all $x\in[a,z]$, and if we choose $x=a$ and $y=z$, we get the following inequality:

$$\begin{aligned}
&g\left(\frac{x+y}{2}\right)\left\{\mathcal{I}^{\beta}_{z^-}w(a)+\mathcal{I}^{\beta}_{a^+}w(z)\right\}\le\mathcal{I}^{\beta}_{z^-}\mathrm{gw}(a)+\mathcal{I}^{\beta}_{a^+}\mathrm{gw}(z)\\
&\le\frac{g(a)+g(z)}{2}\left\{\mathcal{I}^{\beta}_{z^-}w(a)+\mathcal{I}^{\beta}_{a^+}w(z)\right\},
\end{aligned}\tag{89}$$

which was introduced by Iscan [35]

The next result is as follows.

Theorem 9. *Let φ: $\mathsf{F}\longrightarrow[a,z]\subseteq\mathbb{R}$, with $0\le a<z$, be a continuous strictly increasing function, g: $[a,z]\longrightarrow\mathbb{R}$ be a convex differentiable function on (a,z), and w: $[a,z]\longrightarrow\mathbb{R}$ be a nonnegative integrable function satisfying that $(g^{\circ}\varphi)$, $(w^{\circ}\varphi)$: $\mathsf{F}\longrightarrow\mathbb{R}$ are integrable mappings on F. Then, we have*

$$\begin{aligned}
&g\left(a+z-\frac{x+y}{2}\right)^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\\
&\le\frac{1}{2}\Big\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^-}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\\
&\quad+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^+}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\Big\}\\
&\le(g(a)+g(z))^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^+}(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)\\
&\quad-\frac{1}{2}\Big\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^+}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)\\
&\quad+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^-}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\Big\}\\
&\le\left\{g(a)+g(z)-g\left(\frac{x+y}{2}\right)\right\}^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^+}(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right).
\end{aligned}\tag{90}$$

Proof. The first inequality in (90) is already proved in Theorem 8. To prove the second inequality, we have by Lemma (1) according to the convexity of the function g on $[a,z]$ what follows

$$g(a+z-(\eta x+(1-\eta)y)) \leq g(a)+g(z)-g(\eta x+(1-\eta)y), \tag{91}$$

$$g(a+z-(\eta y+(1-\eta)x)) \leq g(a)+g(z) - g(\eta y+(1-\eta)x). \tag{92}$$

Adding inequalities (91) and (92), we get

$$\begin{aligned} &g(a+z-(\eta x+(1-\eta)y))+g(a+z-(\eta y+(1-\eta)x)) \\ &\quad \leq 2g(a)+2g(z)-\{g(\eta x+(1-\eta)y)+g(\eta y+(1-\eta)x)\}. \end{aligned} \tag{93}$$

On both sides of (93), taking product by $1/2h(a+z-(\eta x+(1-\eta)y))$, integrating the estimating inequality with respect to η over $[0,1]$, and then applying the change of variables $\varphi(u)=a+z-(\eta x+(1-\eta)y)$, to the left-hand side, and applying the change of variables $\varphi(u)=\eta x+(1-\eta)y$, to the right-hand side, we get

$$\begin{aligned} &\frac{1}{2}\left\{\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)} (g^{\circ}\varphi)(u)(h^{\circ}\varphi)(u)\frac{\varphi'(u)}{y-x}\mathrm{d}u \right.\\ &\quad \left. +\int_{\varphi^{-1}(a+z-y)}^{\varphi^{-1}(a+z-x)} (g^{\circ}\varphi)(u)h(2(a+z)-(x+y+\varphi(u)))\frac{\varphi'(u)}{y-x}\mathrm{d}u\right\} \\ &\leq (g(a)+g(z))\int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)} h(a+z-\varphi(u))\frac{\varphi'(u)}{y-x}\mathrm{d}u \\ &\quad -\frac{1}{2}\left\{\int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)} (g^{\circ}\varphi)(u)h(a+z-\varphi(u))\frac{\varphi'(u)}{y-x}\mathrm{d}u \right.\\ &\quad \left. +\int_{\varphi^{-1}(x)}^{\varphi^{-1}(y)} (g^{\circ}\varphi)(u)h(a+z-x-y+\varphi(u))\frac{\varphi'(u)}{y-x}\mathrm{d}u\right\}. \end{aligned} \tag{94}$$

Choosing

$$\begin{aligned} (h^{\circ}\varphi)(u) &= \frac{1}{\delta^{\beta}\Gamma(\beta)}\exp\left[\frac{\delta-1}{\delta}(\varphi(u)-\{a+z-y\})\right] \\ &\quad \cdot (\varphi(u)-\{a+z-y\})^{\beta-1}(w^{\circ}\varphi)(u), \end{aligned} \tag{95}$$

then (94) leads to

$$\begin{aligned} &\frac{1}{2(y-x)}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right.\\ &\quad \left. +\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\} \\ &\leq \frac{(g(a)+g(z))^{\varphi}}{y-x}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right) \\ &\quad -\frac{1}{2(y-x)}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)\right.\\ &\quad \left. +\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\}, \end{aligned} \tag{96}$$

which proves the second inequality in (90). To prove the last inequality, by using the convexity of g, we have, for all $x, y \in [a, z]$,

$$g\left(\frac{x+y}{2}\right) = g\left(\frac{(\eta x + (1-\eta)y) + (\eta y + (1-\eta)x)}{2}\right) \leq \frac{1}{2} g(\eta x + (1-\eta)y) + g(\eta y + (1-\eta)x), \tag{97}$$

which can be rewritten as

$$g(a) + g(z) - \frac{1}{2}\{g(\eta x + (1-\eta)y) + g(\eta y + (1-\eta)x)\} \leq g(a) + g(z) - g\left(\frac{x+y}{2}\right). \tag{98}$$

On both sides of (98), taking product by $h(a + z - (\eta x + (1-\eta)y))$ and integrating the estimating inequality with respect to η over $[0,1]$,

$$\begin{aligned}&[g(a) + g(z)]\int_0^1 h(a+z-(\eta x + (1-\eta)y))d\eta \\ &-\frac{1}{2}\left\{\int_0^1 g(\eta x + (1-\eta)y)h(a+z-(\eta x+(1-\eta)y))d\eta\right. \\ &\left.+\int_0^1 g(\eta y + (1-\eta)x)h(a+z-(\eta x+(1-\eta)y))d\eta\right\} \\ &\leq \left\{g(a) + g(z) - g\left(\frac{x+y}{2}\right)\right\}\int_0^1 h(a+z-(\eta x+(1-\eta)y))d\eta.\end{aligned} \tag{99}$$

Applying the change of variables $\varphi(u) = a + z - (\eta x + (1-\eta)y)$, to the right-hand side and applying the same process to the left-hand side as above, we get

$$\begin{aligned}&\frac{(g(a)+g(z))^{\varphi}}{y-x}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right) \\ &-\frac{1}{2(y-x)}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)\right. \\ &\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\} \\ &\leq \left\{g(a)+g(z)-g\left(\frac{x+y}{2}\right)\right\}^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right),\end{aligned} \tag{100}$$

which proves the last inequality in (90). Hence, the proof is thus completed. □

Corollary 2. *Let* φ: F $\longrightarrow [a,z] \subseteq \mathbb{R}$, *with* $0 \leq a < z$, *be a continuous strictly increasing function,* g: $[a,z] \longrightarrow \mathbb{R}$ *be a* *convex differentiable function on* (a, z), *and* $w\colon [a, z] \longrightarrow \mathbb{R}$ *be a nonnegative integrable function satisfying that* $(g^{\circ}\varphi)$, $(w^{\circ}\varphi)$: $\digamma \longrightarrow \mathbb{R}$ *are integrable mappings on* $\digamma$. *Then, we have*

$$\begin{aligned}
&g\left(a+z-\frac{x+y}{2}\right)^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\\
&\leq\frac{1}{2}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right.\\
&\quad\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\}\\
&\leq(g(a)+g(z))^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\\
&\quad-\frac{1}{2}\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)\right.\\
&\quad\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\}\\
&\leq\left\{g(a)+g(z)-g\left(\frac{x+y}{2}\right)\right\}^{\varphi}\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right).
\end{aligned}\tag{101}$$

Proof. This corollary can be easily demonstrated by following the proof of Theorem 9, taking

$$\begin{aligned}
(h^{\circ}\varphi)(u)&=\frac{1}{\delta^{\beta}\Gamma(\beta)}\exp\left[\frac{\delta-1}{\delta}(\{a+z-x\}-\varphi(u))\right]\\
&\quad\cdot(\{a+z-x\}-\varphi(u))^{\beta-1}(w^{\circ}\varphi)(u).
\end{aligned}\tag{102}$$

□

Remark 4. By adding inequalities (90) and (101), we can derive the following inequality:

$$\begin{aligned}
&g\left(a+z-\frac{x+y}{2}\right)\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right.\\
&\quad\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\}\\
&\leq\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-x)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-y)\right)\right.\\
&\quad\left.+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(a+z-y)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(a+z-x)\right)\right\}\\
&\leq(g(a)+g(z))\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\}\\
&\quad-\left\{\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(g^{\circ}\varphi)(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right\}\\
&\leq\left\{g(a)+g(z)-g\left(\frac{x+y}{2}\right)\right\}\left(\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(x)\}^{+}}(w^{\circ}\varphi)\left(\varphi^{-1}(y)\right)+\varphi\mathcal{I}^{\beta,\delta}_{\{\varphi^{-1}(y)\}^{-}}(w^{\circ}\varphi)\left(\varphi^{-1}(x)\right)\right).
\end{aligned}\tag{103}$$

Remark 5

Putting $\delta=1$, we obtain the proportional fractional integral version of each of Theorem 9, Corollary 2, and inequality (103) with respect to the positive increasing function φ

By taking $\varphi(x)=x$, for all $x\in[a,z]$, we obtain the proportional fractional integral version for each of Theorem 9, Corollary 2, and inequality (103).

If we put $\delta=1$ and $\varphi(x)=x$, for all $x\in[a,z]$, we obtain inequality (10), for Riemann–Liouville fractional integral introduced by Iscan [20].

5. Conclusion

In view of the significant importance recently achieved by fractional calculus and its very important applications in the interpretation and modeling of natural phenomena, it has become necessary to develop and refine our capabilities to generalize some of the recent results related to this topic. We achieved our goals of introducing a new fractional Hermite–Hadamard–Mercer's inequality and its fractional integral type inequalities by employing the proportional fractional operators of integrable functions with respect to another continuous and strictly increasing function. We enhanced our work by establishing some new fractional weighted φ-proportional fractional integral Hermite–Hadamard–Mercer type inequalities. Also, in this article, we were keen to present some special cases related to our current study compared to the previous work of the inequality under study. In future work, we recommend researchers study the current inequality via recent fractional operators such as the Atangana + Baleanu operator or Caputo + Fabrizio operator.

Data Availability

The data analysis in this article is all theory.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies. Elsevier, Amsterdam, 2006.

[2] S. G. Samko, A. A. Kilbas, O. I. Marichev et al., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon & Breach, Yverdon, Switzerland, 1993.

[3] A. Atangana and D. Baleanu, "New fractional derivative with non-local and non-singular kernel," *Thermal Science*, vol. 20, no. 2, pp. 757–763, 2016.

[4] A. Atangana and I. Koca, "Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order," *Chaos, Solitons & Fractals*, vol. 89, pp. 447–454, 2016.

[5] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, Springer, New York, US, 2006.

[6] E. F. Beckenbach, "Generalized convex functions," *Bulletin of the American Mathematical Society*, vol. 43, no. 6, pp. 363–371, 1937.

[7] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[8] A. El Farissi, Z. Latreuch, and B. Belaidi, "Hadamard-type inequalities for twice differentiable functions," *RGMIA*, vol. 12, no. 1, p. 7, 2009.

[9] A. Florea and C. P. Niculescu, "A Hermite+Hadamard inequality for convex-concave symmetric functions," *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, vol. 50, no. 2, pp. 149–156, 2007.

[10] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite+-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, pp. 2403–2407, 2013.

[11] K. Liu, J. Wang, and D. O'Regan, "On the Hermite+Hadamard type inequality for ψ-Riemann+Liouville fractional integrals via convex functions," *Journal of Inequalities and Applications*, vol. 27, 2019.

[12] T. A. Aljaaidi, D. B. Pachpatte, T. Abdeljawad, M. S. Abdo, M. A. Almalahi, and S. S. Redhwan, "Generalized proportional fractional integral Hermite-Hadamard's inequalities," *Advances in Difference Equations*, vol. 2021, no. 1, p. 493, 2021.

[13] A. McD Mercer, "A variant of Jensen's inequality," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, p. 73, 2003.

[14] H. R. Moradi and S. Furuichi, "Improvement and generalization of some Jensen-Mercer-type inequalities," *Journal of Mathematical Inequalities*, vol. 14, no. 2, pp. 377–383, 2020.

[15] M. A. Khan, Z. Husain, and Y. M. Chu, "New estimates for Csiszár divergence and zipf+mandelbrot entropy via jensen+mercer's inequality," *Complexity*, vol. 2020, Article ID 8928691, 8 pages, 2020.

[16] A. Markova, J. Pecaric, and I. Peric, "A variant of Jensens inequality of Mercer's type for operators with applications," *Linear Algebra and its Applications*, vol. 418, pp. 551–564, 2006.

[17] M. Niezgoda, "A generalization of Mercer's result on convex functions," *Nonlinear Analysis Forum*, vol. 71, 2009.

[18] E. Anjidani and M. R. Changalvaiy, "Reverse jensen-mercertype operator inequalities," *Fe Electronic Journal of Linear Algebra*, vol. 31, pp. 87–99, 2016.

[19] H. Ogulmus and M. Sarikaya, "Hermite-Hadamard-Mecer type inequalities for fractional integrals," 2021, https://www.pmf.ni.ac.rs/filomat.

[20] I. Iscan, "Weighted Hermite+Hadamard+Mercer type inequalities for convex functions," *Numerical Methods for Partial Differential Equations*, vol. 37, pp. 118–130, 2021.

[21] T. Abdeljawad, M. A. Ali, P. O. Mohammed, and A. Kashuri, "On inequalities of Hermite+Hadamard+Mercer type involving Riemann-Liouville fractional integrals," *AIMS Math*, vol. 6, no. 1, pp. 712–725, 2020.

[22] S. I. Butt, M. Nadeem, S. Qaisar, A. O. Akdemir, and T. Abdeljawat, "Hermite+Jensen+Mercer type inequalities for conformable integrals and related results," *Adv Differ Equ*, vol. 50, 2020.

[23] H.-H. Chu, S. Rashid, Z. Hammouch, and Y.-M. Chu, "New fractional estimates for Hermite-Hadamard-Mercer's type inequalities," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3079–3089, 2020.

[24] M. Vivas-Cortez, M. A. Ali, A. Kashuri, and H. Budak, "Generalizations of fractional Hermite-Hadamard-Mercer like inequalities for convex functions," *AIMS Math*, vol. 6, no. 9, pp. 9397–9421, 2021.

[25] S. I. Butt, S. Yousaf, A. Asghar, K. A. Khan, and H. R. Moradi, "New fractional Hermite+Hadamard+Mercer inequalities for harmonically convex function," *Journal of Function Spaces*, vol. 2021, Article ID 5868326, 11 pages, 2021.

[26] M. Khan, S. Anwar, S. Khalid, and Z. Sayed, "Inequalities of the type Hermite+Hadamard+Jensen+Mercer for strong convexity," *Mathematical Problems in Engineering*, vol. 2021, Article ID 5386488, 16 pages, 2021.

[27] E. Set, B. Celik, M. E. Ozdemir, and M. Aslan, "Some new results on Hermite+Hadamard+Mercer-type inequalities using a general family of fractional integral operators," *Fractal Fract*, vol. 5, p. 68, 2021.

[28] T. A. Aljaaidi and D. Pachpatte, “New generalization of reverse Minkowski’s inequality for fractional integral,” *Adv. Theory. Nonlinear Anal. Appl.*vol. 1, pp. 72–81, 2021.
[29] F. Jarad, T. Abdeljawad, and J. Alzabut, “Generalized fractional derivatives generated by a class of local proportional derivatives,” *The European Physical Journal - Special Topics*, vol. 226, pp. 3457–3471, 2017.
[30] F. Jarad, M. A. Alqudah, and T. Abdeljawad, “On more generalized form of proportional fractional operators,” *Open Mathematics*, vol. 18, pp. 167–176, 2020.
[31] T. A. Aljaaidi, D. B. Pachpatte, W. Shatanawi, M. S. Abdo, and K. Abodayeh, “Generalized proportional fractional integral functional bounds in Minkowski’s inequalities,” *Adv Differ Equ*, vol. 419, 2021.
[32] K. Yildirim and S. Yildirim, “On the hermite-hadamard-mercer type inequalities for generalized proportional fractional integrals,” *Preprints*, vol. 2020, p. 2020120687.
[33] S. I. Butt, A. Kashuri, M. Umar, and A. Aslam, “Wei Gao, Hermite-Jensen-Mercer type inequalities via psi-Riemann-Liouville k-fractional integrals,” *AIMS Math*, vol. 5, no. 5, pp. 5193–5220, 2020.
[34] M. Kian and M. S. Moslehian, “Refinements of the operator Jensen-Mercer inequality,” *The Electronic Journal of Linear Algebra*, vol. 26, pp. 742–753, 2013.
[35] I. Iscan, “Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals,” *Studia Universitatis Babes-Bolyai Mathematica*, vol. 60, no. 3, pp. 355–366, 2015.