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Research Article

# Properties of Some Dynamic Partial Integrodifferential Equations on Time Scales

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**Abstract.** The main objective of this paper is to study some properties of new dynamic equations partial integrodifferential equations in two independent variables on time scales. The tools used in obtaining our results are application of Banach Fixed point theorem. We also used dynamic inequality with explicit estimates to study the properties of solution of dynamic partial integrodifferential equation.

**Keywords.** Explicit estimate; Integral inequality; Two variables; Time scales

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## 1. Introduction

In the recent years abundance of applications are stimulating the rapid development of dynamic equations on time scales. This has attracted and continuous to attract the considerable attention in the literature. The theory of time scale calculus was firstly initiated in 1990 by the German Mathematician Steafan Hilger [7] in his Ph.d dissertation. Recently, many authors have obtained results on various types of dynamic equations and its applications on time scales [1, 2]. In [6, 8] properties of some Voltera type equations are studied. Stability and convergence of dynamical systems is studied by authors in [9, 10]. Motivated by the scope of dynamic equations in this paper we have established some new dynamic integrodifferntial equations in two independent variables.

Let $\mathbb{R}^n$ denotes the real $n$-dimensional Euclidean space with appropriate norm $|\cdot|$. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales and $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$. The set of all $rd$-continuous function is denoted by $C_{rd}$. The partial delta derivaive of $p(x,y)$ for $(x,y)$ with respect to $x,y$ and $xy$ is denoted by $p^{\Delta_1}(x,y)$, $p^{\Delta_2}(x,y)$ and $p^{\Delta_1\Delta_2}(x,y) = p^{\Delta_2\Delta_1}(x,y)$.

In this paper, we study the dynamic integrodifferential equation of the form

$$u^{\Delta_1\Delta_2}(x,y) = f(x,y,u(x,y),(Gu)(x,y),(Hu)(x,y)), \tag{1.1}$$

with the given conditions

$$u(x,x_0) = \alpha(x), u(y,y_0) = \beta(y), u(x_0,y_0) = 0, \tag{1.2}$$

for $(x,y) \in \Omega$ where

$$(Gu)(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} g(x,y,s,t,u(s,t))\Delta t\Delta s, \tag{1.3}$$

$$(Hu)(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} h(x,y,s,t,u(s,t))\Delta t\Delta s, \tag{1.4}$$

for $f \in C_{rd}(\Omega\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $g \in C_{rd}(\Omega\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $\alpha,\beta \in C_{rd}(\mathbb{T},\mathbb{R}^n)$, $(G0)(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} g(x,y,s,t,0)\Delta s\Delta t$ and $(H0)(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} h(x,y,s,t,0)\Delta t\Delta s$.

## 2. Preliminaries

Let $G$ be a space of functions which satisfy the condition

$$|\Phi(x,y)|_1 \le O(e_\lambda(x,y)), \quad \text{for } (x,y) \in \Omega, \tag{2.1}$$

where $\lambda > 0$ is a constant. In space $G$, we define the norm

$$|\Phi|_G = \sup_{(x,y)\in\Omega} [\||\Phi(x,y)|_1 e_{\ominus\lambda}(x,y)\|]. \tag{2.2}$$

The norm defined in (2.2) is a Banach Space. Then from condition (2.1) implies there exists a condition $N \ge 0$ such that

$$|\Phi(x,y)|_G \le N e_\lambda(x,y). \tag{2.3}$$

From (2.3), we have

$$|\Phi|_G \le N. \tag{2.4}$$

It is easy to see that $u(x,y)$ is solution of (1.1)-(1.2) if it satisfies the equation

$$u(x,y) = \alpha(x) + \beta(y) + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t,u(s,t),(Gu)(s,t),(Hu)(s,t))\Delta s\Delta t. \tag{2.5}$$

Now, we give the lemma proved in [5].

**Lemma 2.1.** *Let*

$$u^{\Delta}(t) \le a(t)u(t), \quad \textit{for all } t \in T^k, \tag{2.6}$$

*then*

$$u(t) \le u(t_0)e_a(t,t_0), \quad \textit{for all } t \in T^k. \tag{2.7}$$

The following theorem is proved in [12].

**Lemma 2.2.** *Let* $u,a\in C_{rd}(\Omega,\mathbb{R})$, $g(x,y,s,t)\in(\Omega\times\Omega,\mathbb{R}_+)$, $h(x,y,x,t,\xi,\tau)\in(\Omega\times\Omega\times\Omega,\mathbb{R}_+)$ *and* $k\geq 0$ *is a constant. If*

$$u(x,y)\leq k+\int_{x_0}^{x}\int_{y_0}^{y}a(s,t)u(s,t)\Delta t\Delta s+\int_{x_0}^{x}\int_{y_0}^{y}\left(\int_{s_0}^{s}\int_{t_0}^{t}g(s,t,\xi,\tau)u(\xi,\tau)\Delta\tau\Delta\xi\right)\Delta t\Delta s$$
$$+\int_{x_0}^{x}\int_{y_0}^{y}\left(\int_{s_0}^{s}\int_{t_0}^{t}(h(s,t,\xi,\tau,m,n)u(m,n)\Delta n\Delta m)\Delta\tau\Delta\xi\right)\Delta t\Delta s, \tag{2.8}$$

*for* $(x,y)\in\Omega$ *then*

$$u(x,y)\leq ke_{A(x,y)}(x,x_0), \tag{2.9}$$

*where*

$$A(x,y)=a(x,y)u(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}g(x,y,\xi,\tau)u(\xi,\tau)\Delta\tau\Delta\xi$$
$$+\int_{x_0}^{x}\int_{y_0}^{y}\left(\int_{s_0}^{s}\int_{t_0}^{t}h(x,y,\xi,\tau,m,n)u(m,n)\Delta n\Delta m\right). \tag{2.10}$$

Now, we prove the inequality which is useful in studying the properties of solution of (1.1)-(1.2).

**Theorem 2.3.** *Let* $u,p,q,r\in C_{rd}(\Omega,\mathbb{R}_+)$ *and*

$$u(x,y)\leq k+\int_{x_0}^{x}\int_{y_0}^{y}p(s,t)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}q(\xi,\tau)u(\xi,\tau)\Delta\tau\Delta\xi\right.$$
$$\left.+\int_{x_0}^{a}\int_{y_0}^{b}r(\xi,\tau)u(\xi,\tau)\Delta\tau\Delta\xi\right]\Delta t\Delta s,\ \textit{for } (x,y)\in\Omega, \tag{2.11}$$

*where* $k\geq 0$ *is a constant. If*

$$c=\int_{x_0}^{a}\int_{y_0}^{b}r(\xi,\tau)e_{\int_{x_0}^{x}[p(x,t)+q(x,t)]\Delta t}(x,x_0)<1, \tag{2.12}$$

*then*

$$u(x,y)\leq\frac{k}{1-c}e_{\int_{x_0}^{x}[p(x,t)+q(x,t)]\Delta t}(x,x_0). \tag{2.13}$$

*Proof.* Define a function $v(x,y)$ by right hand side of (2.13) $v(x_0,y)=v(x,y_0)=c$, $u(x,y)\leq v(x,y)$.

$$v^{\Delta_2\Delta_1}(x,y)=p(x,y)\left[u(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}q(s,t)u(s,t)\Delta t\Delta s+\int_{x_0}^{a}\int_{y_0}^{b}r(s,t)u(s,t)\Delta t\Delta s\right]$$
$$\leq p(x,y)\left[v(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}q(s,t)v(s,t)\Delta t\Delta s+\int_{x_0}^{a}\int_{y_0}^{b}r(s,t)v(s,t)\Delta t\Delta s\right]. \tag{2.14}$$

Define a function $f(x,y)$ by

$$f(x,y)=v(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}q(s,t)v(s,t)\Delta t\Delta s+\int_{x_0}^{a}\int_{y_0}^{b}r(s,t)v(s,t)\Delta t\Delta s. \tag{2.15}$$

We get

$$f(x_0,y)=f(x,y_0)$$

$$= c + \int_{x_0}^{a} \int_{y_0}^{b} r(s,t)v(s,t)\Delta t \Delta s$$
$$= M \quad \text{(say)}. \tag{2.16}$$

We have

$$\begin{aligned} f^{\Delta_2\Delta_1}(x,y) &= v^{\Delta_2\Delta_1}(x,y) + q(x,y)v(x,y) \\ &\le p(x,y)v(x,y) + q(x,y)v(x,y) \\ &\le p(x,y)f(x,y) + q(x,y)f(x,y) \\ &= f(x,y)[p(x,y) + q(x,y)]. \end{aligned} \tag{2.17}$$

By keeping $x$ fixed in (2.17) and $y = t$ and delta integrating from $y_0$ to $y$, $f^{\Delta_1}(x,y) = 0$ and $f(x,y)$ is non decreasing. We have

$$\begin{aligned} f^{\Delta_1}(x,y) &\le \int_{y_0}^{y} [p(x,t) + q(x,t)]f(x,t)\Delta t \\ &\le f(x,y) \int_{y_0}^{y} [p(x,t) + q(x,t)]\Delta t. \end{aligned} \tag{2.18}$$

Let $Q(x,y) = \int_{y_0}^{y} [p(x,t) + q(x,t)]\Delta t$, then from (2.18)

$$f^{\Delta_1}(x,y) \le f(x,y)Q(x,y). \tag{2.19}$$

Now treating $y$ fixed and applying Lemma 2.1, we have

$$f(x,y) \le M e_{Q(x,y)}(x,x_0). \tag{2.20}$$

From (2.18) and (2.20), we have

$$M \le \frac{k}{1-c}. \tag{2.21}$$

Using (2.20) and (2.21) and the fact that $v(x,y) \le f(x,y)$, we get (2.13). □

## 3. Existence and Uniqueness

Now, we give existence and uniqueness of (1.1)-(1.2).

**Theorem 3.1.** *Suppose the function $f$, $g$, $h$ in (1.1)-(1.4) satisfies the conditions:*

$$|f(x,y,u,v,w) - f(x,y,\overline{u},\overline{v},\overline{w})| \le k(x,y)[|u-\overline{u}| + |v-\overline{v}| + |w-\overline{w}|], \tag{3.1}$$
$$|g(x,y,s,t,u) - g(x,y,s,t,\overline{u})| \le l(x,y,s,t)[|u-\overline{u}|], \tag{3.2}$$
$$|h(x,y,s,t,v) - h(x,y,s,t,\overline{v})| \le m(x,y,s,t)[|v-\overline{v}|], \tag{3.3}$$

*where $k(x,y) \in C_{rd}(\Omega,\mathbb{R})$, $m(x,y,s,t), l(x,y,s,t) \in C_{rd}(\Omega \times \Omega,\mathbb{R})$.*
*For $\lambda$ as in (2.1)*

(i) *there exists a non negative constant $c < 1$ such that*

$$Q(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} Q(s,t)\Delta t \Delta s \le c e_\lambda(x,y), \tag{3.4}$$

$$Q(x,y) = k(x,y)\Bigg[e_\lambda(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} l(x,y,s,t)e_\lambda(s,t)\Delta s \Delta t$$

$$+\int_{x_0}^{a}\int_{y_0}^{b} m(x,y,s,t)e_\lambda(s,t)\Delta s\Delta t\Bigg], \tag{3.5}$$

(ii) *there exists non-negative constant* $\eta$ *such that*

$$|\alpha(x)|+|\beta(y)|+|f(x,y,u(x_0,y_0),(G0)(x,y),(H0)(x,y))|$$
$$+\int_{x_0}^{x}\int_{y_0}^{y} f(x,y,u(x_0,y_0),(G0)(x,y),(H0)(x,y))e_\lambda(s,t)\Delta s\Delta t, \tag{3.6}$$

*then the initial boundary value problem* (1.1)-(1.2) *has a unique solution on* $\Omega$.

*Proof.* Let $u\in G$ and we define the operator $S$ by

$$(Su)(x,y)=|\alpha(x)|+|\beta(y)|+\int_{x_0}^{x}\int_{y_0}^{y} f(x,y,u(s,t),(Gu)(x,y),(Hu)(x,y))\Delta s\Delta t. \tag{3.7}$$

By delta differentiating (3.7) with respect to $x$ and $y$ we have

$$(Su)^{\Delta_2\Delta_1}(x,y)=\int_{x_0}^{x}\int_{y_0}^{y} f(x,y,u(s,t),(Gu)(x,y),(Hu)(x,y))\Delta s\Delta t. \tag{3.8}$$

We show that $Su$ maps $G$ into itself and $Su$ is rd-continuous on $\Omega$

$$
\begin{aligned}
&|(Su)(x,y)|+\left|(Su)^{\Delta_2\Delta_1}(x,y)\right|\\
&\le \alpha(x)+\beta(y)+\int_{x_0}^{x}\int_{y_0}^{y}|f(s,t,u(s,t),(Gu)(s,t),(Hu)(s,t))-f(s,t,u(x_0,y_0),(G0)(s,t),(H0)(s,t))|\\
&\quad+\int_{x_0}^{x}\int_{y_0}^{y}|f(s,t,u(x_0,y_0),(G0)(s,t),(H0)(s,t))|\Delta t\Delta s+|f(x,y,u(x,y),(Gu)(x,y),(Hu)(x,y))\\
&\quad-f(x,y,u(x_0,y_0),(G0)(x,y),(H0)(x,y))|+|f(x,y,u(x_0,y_0),(G0)(x,y),(H0)(x,y))|\\
&\le \eta e_\lambda(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}k(s,t)\Big\{\left[|u(s,t)|_1\right]+\int_{x_0}^{s}\int_{y_0}^{t}l(s,t,m,n)\left[|u(m,n)|_1\right]\Delta n\Delta m\\
&\quad+\int_{x_0}^{a}\int_{y_0}^{b}m(s,t,m,n)\left[|u(m,n)|_1\right]\Delta t\Delta s\Big\}\Delta y\Delta x\\
&\quad+k(x,y)\left[|u(x,y)|_1+\int_{x_0}^{x}\int_{y_0}^{y}l(x,y,s,t)|u(s,t)|_1\Delta t\Delta s+\int_{x_0}^{a}\int_{y_0}^{b}m(x,y,s,t)|u(s,t)|_1\Delta t\Delta s\right]\\
&\le \eta e_\lambda(x,y)+|u|_G\{k(x,y)\Big[e_\lambda(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}l(x,y,s,t)e_\lambda(s,t)\Delta t\Delta s\\
&\quad+\int_{x_0}^{a}\int_{y_0}^{b}m(x,y,s,t)e_\lambda(s,t)\Delta t\Delta s\Big]\Big\}\\
&\quad+\int_{x_0}^{x}\int_{y_0}^{y}\{k(s,t)\Big[e_\lambda(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}l(s,t,m,n)e_\lambda(m,n)\Delta n\Delta m\\
&\quad+\int_{x_0}^{a}\int_{y_0}^{b}m(s,t,m,n)e_\lambda(m,n)\Delta n\Delta m\Big]\Big\}\Delta t\Delta s\\
&\le \eta e_\lambda(x,y)+N\left\{Q(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}Q(s,t)\Delta t\Delta s\right\}\\
&\le [\eta+Nc]\,e_\lambda(x,y).
\end{aligned} \tag{3.9}
$$

From (3.9) it follows that $Su\in G$ which proves that $S$ maps $G$ into itself.

Now, we verify that $S$ is a contraction map. Let $u(x,y), v(x,y) \in G$.

From (3.7) and (3.8), we have

$$\begin{aligned}
&|(Su)(x,y)-(Sv)(x,y)| + \left|(Su)^{\Delta_2\Delta_1}(x,y)-(Sv)^{\Delta_2\Delta_1}(x,y)\right| \\
&\le \int_{x_0}^{x}\int_{y_0}^{y} |f(s,t,u(s,t),(Gu)(s,t),(Hu)(s,t)) - f(s,t,v(s,t),(Gv)(s,t),(Hv)(s,t))|\,\Delta t\Delta s \\
&\quad + |f(x,y,u(x,y),(Gu)(x,y),(Hu)(x,y)) - f(x,y,v(x,y),(Gv)(x,y),(Hv)(x,y))| \\
&\le \int_{x_0}^{x}\int_{y_0}^{y} \Big\{k(s,t)\left[|u(s,t)-v(s,t)|_1\right] + \int_{x_0}^{s}\int_{y_0}^{t} l(s,t,m,n)\left[|u(m,n)-v(m,n)|_1\right]\Delta n\Delta m \\
&\quad + \int_{x_0}^{a}\int_{y_0}^{b} m(s,t,m,n)\left[|u(m,n)-v(m,n)|_1\right]\Delta n\Delta m\Big\}\Delta t\Delta s + k(x,y)\left\{|u(x,y)-v(x,y)|_G\right. \\
&\quad \left.+ \int_{x_0}^{x}\int_{y_0}^{y} l(x,y,s,t)\left[|u(s,t)-v(s,t)|_1\right]\Delta t\Delta s + \int_{x_0}^{a}\int_{y_0}^{b} m(x,y,s,t)\left[|u(s,t)-v(s,t)|_1\right]\Delta t\Delta s\right\} \\
&\le |u-v|_G k(x,y)\Big\{e_\lambda(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} l(x,y,s,t)e_\lambda(s,t)\Delta t\Delta s + \int_{x_0}^{a}\int_{y_0}^{b} m(x,y,s,t)e_\lambda(s,t)\Delta t\Delta s\Big\} \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\Big\{e_\lambda(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} l(s,t,m,n)e_\lambda(s,t)\Delta n\Delta m \\
&\quad + \int_{x_0}^{a}\int_{y_0}^{b} m(s,t,m,n)e_\lambda(s,t)\Delta n\Delta m\Big\}\Delta t\Delta s \\
&= |u-v|_G\left\{Q(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t)\Delta t\Delta s\right\} \\
&\le c|u-v|_G e_\lambda(x,y).
\end{aligned} \tag{3.10}$$

From (3.10), we have

$$|Su-Sv|_G \le c|u-v|_G. \tag{3.11}$$

Since $c<1$, it follows from Banach Fixed point Theorem $S$ has a unique fixed point in $G$. The fixed point of $G$ is a solution of (1.1) and (1.2). This completes the proof. □

**Theorem 3.2.** *Suppose function $f$ satisfies* (3.1)*, and $g,h$ satisfies the condition*

$$|g(x,y,s,t,u)-g(x,y,s,t,\overline{u})| \le l_1(x,y)m_1(s,t)[|u-\overline{u}|], \tag{3.12}$$

$$|h(x,y,s,t,u)-h(x,y,s,t,\overline{u})| \le l_2(x,y)m_2(s,t)[|u-\overline{u}|], \tag{3.13}$$

*where $l_1,l_2,m_1,m_2 \in C_{rd}(\Omega,\mathbb{R})$. Let $q(x,y)=\max\{k(x,y),l_1(x,y),l_2(x,y)\}$. Then initial boundary value problem* (1.1)-(1.2) *has at most one solution on $\Omega$.*

*Proof.* Let $u_1(x,y)$ and $u_2(x,y)$ be any two solutions of (1.1)-(1.2) and $w(x,y)=|u_1(x,y)-u_2(x,y)|$ then by hypotheses, we have

$$\begin{aligned}
w(x,y) &\le \int_{x_0}^{x}\int_{y_0}^{y} |f(s,t,(Gu_1)(s,t),(Hu_1)(s,t)) - f(s,t,(Gu_2)(s,t),(Hu_2)(s,t))|\,\Delta t\Delta s \\
&\le \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\Big[|u_1(s,t)-u_2(s,t)| + \int_{x_0}^{s}\int_{y_0}^{t} l_1(\xi,\tau)m_1(\xi,\tau)|u_1(\xi,\tau)-u_2(\xi,\tau)|\Delta\tau\Delta\xi
\end{aligned}$$

$$+\int_{x_0}^{a}\int_{y_0}^{b} l_2(\xi,\tau)m_2(\xi,\tau)|u_1(\xi,\tau)-u_2(\xi,\tau)|\Delta\tau\Delta\xi\Big]\Delta t\Delta s$$

$$\le \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)\Big[|u_1(s,t)-u_2(s,t)|+\int_{x_0}^{s}\int_{y_0}^{t} m_1(\xi,\tau)|u_1(\xi,\tau)-u_2(\xi,\tau)|\Delta\tau\Delta\xi$$

$$+\int_{x_0}^{a}\int_{y_0}^{b} m_2(\xi,\tau)|u_1(\xi,\tau)-u_2(\xi,\tau)|\Delta\tau\Delta\xi\Big]\Delta t\Delta s. \tag{3.14}$$

Applying Lemma 2.2 to (3.14), where $k=0$, we have

$$|u_1(x,y)-u_2(x,y)|\le 0, \tag{3.15}$$

which implies that $u_1(x,y)=u_2(x,y)$ for $(x,y)\in\Omega$. Therefore, there is atmost one solution of (1.1) and (1.2) on $\Omega$. □

## 4. Properties of Solution

Let $u(x,y)\in C_{rd}(\Omega,\mathbb{R}_+)$, $u^{\Delta_2\Delta_1}(x,y)$ exists and satisfies the inequality

$$\left|u^{\Delta_2\Delta_1}(x,y)-f(x,y,u(x,y),(Gu)(x,y),(Hu)(x,y))\right|\le\epsilon,$$

for a given $\epsilon\ge 0$, where (1.2) holds. Then, we say that $u(x,y)$ is an $\epsilon$-approximate solution of (1.1) with (1.2).

In the following theorem, we obtain the estimate the difference between two approximate solution of (1.1).

**Theorem 4.1.** *Suppose that the functions $f,g,h$ in* (1.1) *satisfy the conditions* (3.1), (3.12) *and* (3.13).
*Let $u_i(x,y)$, $(i=1,2)$ be $\epsilon_i$ approximate solution of* (1.1) *with*

$$u_i(x,x_0)=\alpha_i(x),\ u_i(y_0,y)=\beta_i(y),\ u_i(x_0,y_0)=0, \tag{4.1}$$

*where $\alpha_i,\beta_i\in C_{rd}(\mathbb{T},\mathbb{R}^n)$, such that*

$$\left|\alpha_1(x)-\alpha_2(x)+\beta_1(y)-\beta_2(y)\right|\le\eta, \tag{4.2}$$

*where $\eta\ge 0$ is a constant. Then*

$$|u_1(x,y)-u_2(x,y)|\le\frac{\varepsilon ab+\eta}{1-\overline{c}}e_{\int_{x_0}^{x}[q(x,t)+m_1(x,t)]\Delta t}(x,x_0), \tag{4.3}$$

*where $(x,y)\in\Omega$, and*

$$\overline{c}=\int_{x_0}^{a}\int_{y_0}^{b} m_2(\xi,\tau)e_{\int_{x_0}^{x}[q(x,t)+m_1(x,t)]\Delta t}(x,x_0)<1. \tag{4.4}$$

*Proof.* Since $u_i(x,y)$, $(i=1,2)$ for $(x,y)\in\Omega$ are a $\epsilon$ approximate solution of (1.1) with (4.1), we have

$$\left|u_i^{\Delta_2\Delta_1}(x,y)-f(x,y,u_i(x,y),(Gu_i)(x,y),(Hu_i)(x,y))\right|\le\varepsilon_i. \tag{4.5}$$

Integrating both sides of (4.5) on $\Omega$ and (4.1), we get

$$\epsilon_i xy\ge\int_{x_0}^{x}\int_{y_0}^{y}\left|u_i^{\Delta_2\Delta_1}(s,t)-f(s,t,u_i(s,t),(Gu_i)(s,t),(Hu_i)(s,t))\right|\Delta t\Delta s$$

$$\geq \left| \int_{x_0}^{x} \int_{y_0}^{y} \{u_i^{\Delta_2\Delta_1}(s,t) - f(s,t,u_i(s,t),(Gu_i)(s,t),(Hu_i)(s,t))\} \right| \Delta t \Delta s$$
$$= \left| u_i(x,y) - \alpha_i(x) - \beta_i(y) - \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_i(s,t),(Gu_i)(s,t),(Hu_i)(s,t)) \Delta t \Delta s \right|. \tag{4.6}$$

From (4.6) and using the elementary inequalities

$$|v - z| \leq |v| + |z|, \quad |v| - |z| \leq |v - z|, \tag{4.7}$$

we have

$$(\varepsilon_1 + \varepsilon_2)(xy) \geq \left| u_1(x,y) - [\alpha_1(x) + \beta_1(y)] - \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_1(s,t),(Gu_1)(s,t),(Hu_1)(s,t)) \Delta t \Delta s \right|$$
$$+ \left| u_2(x,y) - [\alpha_2(x) + \beta_2(y)] - \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_2(s,t),(Gu_2)(s,t),(Hu_2)(s,t)) \Delta t \Delta s \right|$$
$$\geq \left| \left\{ |u_1(x,y) - [\alpha_1(x) + \beta_1(y)] - \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_1(s,t),(Gu_1)(s,t),(Hu_1)(s,t)) \Delta t \Delta s \right\} \right.$$
$$\left. - \left\{ u_2(x,y) - [\alpha_2(x) + \beta_2(y)] - \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_2(s,t),(Gu_2)(s,t),(Hu_2)(s,t)) \Delta t \Delta s \right\} \right|$$
$$\geq |u_1(x,y) - u_2(x,y)| - \left| [\alpha_1(x) + \beta_1(y)] + [\alpha_2(x) + \beta_2(y)] \right|$$
$$- \left| \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_1(s,t),(Gu_1)(s,t),(Hu_1)(s,t)) \Delta t \Delta s \right.$$
$$\left. - \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_2(s,t),(Gu_2)(s,t),(Hu_2)(s,t)) \Delta t \Delta s \right|. \tag{4.8}$$

Let $w(x,y) = |u_1(x,y) - u_2(x,y)|$ for $(x,y) \in \Omega$, we get from (4.8)

$$w(x,y) \leq \varepsilon xy + \left| \alpha_1(x) - \alpha_2(x) + \beta_1(y) - \beta_2(y) \right| \int_{x_0}^{x} \int_{y_0}^{y} \left| f(s,t,u_1(s,t),(Gu_1)(s,t),(Hu_1)(s,t)) \right.$$
$$\left. - \int_{x_0}^{x} \int_{y_0}^{y} f(s,t,u_2(s,t),(Gu_2)(s,t),(Hu_2)(s,t)) \right| \Delta t \Delta s$$
$$\leq |\varepsilon st + \eta| + \int_{x_0}^{x} \int_{y_0}^{y} q(s,t) \left[ w(s,t) + \int_{x_0}^{s} \int_{y_0}^{t} m_1(\xi,\tau) w(\xi,\tau) \Delta\tau \Delta\xi \right.$$
$$\left. + \int_{x_0}^{a} \int_{y_0}^{b} m_2(\xi,\tau) w(\xi,\tau) \Delta\tau \Delta\xi \right] \Delta t \Delta s. \tag{4.9}$$

Now an application of Theorem 2.3, we get (4.3). □

**Remark 4.1.** If $u_1(x,y)$ is a solution of (1.1)-(1.2) then we have $\epsilon_1 = 0$ and from (4.3), we have $u_2(x,y) \to u_1(x,y)$ as $\epsilon_2 \to 0$ and $\eta \to 0$. If we have $\epsilon_1 = \epsilon_2 = 0$, $\alpha_1(x) = \alpha_2(x)$, $\beta_1(y) = \beta_2(y)$ in (4.3). Then uniqueness of solution (1.1)-(1.2) can be established.

Now, we consider (1.1)-(1.2) together with the following

$$v^{\Delta_2\Delta_1}(x,y) = \overline{f}(x,y,v(x,y),(Gv)(x,y),(Hv)(x,y)), \tag{4.10}$$
$$v(x,x_0) = \overline{\alpha}(x), \quad v(y_0,y) = \overline{\beta}(y), v(x_0,y_0) = 0, \tag{4.11}$$

where $G,H$ are as defined in (1.3) and (1.4).

Now, we prove the conditions concerning the closeness of solutions of (1.1)-(1.2) and (4.10)-(4.11).

**Theorem 4.2.** *Suppose function $f,g,h$ in satisfy the conditions* (3.12), (3.13), (3.14) *and there exists constant $\overline{\varepsilon} \geq 0$ and $\overline{\eta} \geq 0$ such that*

$$|f(x,y,u,v,w) - \overline{f}(x,y,u,v,w)| \leq \overline{\varepsilon}, \tag{4.12}$$

$$|\alpha(x) - \overline{\alpha}(x) + \beta(y) - \overline{\beta}(y)| \leq \overline{\eta}, \tag{4.13}$$

*where $f,\alpha,\beta$ and $\overline{f},\overline{\alpha},\overline{\beta}$ as in* (1.1)-(1.2) *and* (4.10)-(4.11)*. Let $u(x,y)$ and $v(x,y)$ be respectively the solutions of problem* (1.1)-(1.2) *and* (4.10)-(4.11)*. Then*

$$|u(x,y) - v(x,y)| \leq \frac{\overline{\varepsilon}ab + \overline{\eta}}{1-\overline{c}} e_{\int_{x_0}^{x}[q(x,t)+m_1(x,t)]\Delta t}(x,x_0), \quad \text{for } (x,y) \in \Omega. \tag{4.14}$$

*Proof.* Let $w(x,y) = |u(x,y) - v(x,y)|$ for $(x,y) \in \Omega$. Since $u(x,y)$ and $v(x,y)$ are the solution of problem (1.1)-(1.2) and (4.10)-(4.11), we have

$$\begin{aligned}
w(x,y) &\leq |\alpha(x) - \overline{\alpha}(x) + \beta(y) - \overline{\beta}(y)| + \int_{x_0}^{x}\int_{y_0}^{y} |f(x,y,u(s,t),(Gu)(s,t),(Hu)(s,t)) \\
&\quad - f(x,y,v(s,t),(Gv)(s,t),(Hv)(s,t))|\Delta t\Delta s \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} \Big|f(x,y,v(s,t),(Gv)(s,t),(Hv)(s,t)) \\
&\quad - \overline{f}(x,y,v(s,t),(Gv)(s,t),(Hv)(s,t))\Big|\Delta t\Delta s \\
&\leq \eta + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)\Big[|w(s,t)| + \int_{x_0}^{s}\int_{y_0}^{t} l_1(\xi,\tau)m_1(\xi,\tau)|w(s,t)|\Delta\tau\Delta\xi \\
&\quad + \int_{x_0}^{a}\int_{y_0}^{b} l_2(\xi,\tau)m_2(\xi,\tau)|w(s,t)|\Delta\tau\Delta\xi\Big]\Delta t\Delta s + \int_{x_0}^{x}\int_{y_0}^{y}\overline{\varepsilon}\Delta t\Delta s \\
&\leq \left(\overline{\varepsilon}ab + \overline{\eta}\right) + \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)\Big[|w(s,t)| + \int_{x_0}^{s}\int_{y_0}^{t} m_1(\xi,\tau)|w(s,t)|\Delta\tau\Delta\xi \\
&\quad + \int_{x_0}^{a}\int_{y_0}^{b} m_2(\xi,\tau)|w(s,t)|\Delta\tau\Delta\xi\Big]\Delta t\Delta s.
\end{aligned} \tag{4.15}$$

Now applying Theorem 2.3 to (4.15) we get (4.14). □

**Remark 4.2.** The result given in above Theorem relates the solution of the problem (1.1)-(1.2) and (4.10)-(4.11) in the sense that if $f$ is close to $\overline{f}$, $\alpha$ is close to $\overline{\alpha}$, $\beta$ is close to $\overline{\beta}$ then the solution to problems (1.1)-(1.2) and (4.10)-(4.11) are close together.

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### Competing Interests

The author declares that he has no competing interests.

### Authors' Contributions

The author wrote, read and approved the final manuscript.

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