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Some properties of 2-absorbing primary ideals in lattices

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Abstract

We introduce the concepts of a primary and a 2-absorbing primary ideal and the radical of an ideal in a lattice. We study some properties of these ideals. A characterization for the radical of an ideal to be a primary ideal is given. Also a characterization for an ideal I to be a 2-absorbing primary ideal is proved. Examples and counter examples are given wherever necessary.

Previous Next

Keywords

Ideal; Prime ideal; Primary ideal; 2-absorbing ideal; 2-absorbing primary ideal

1. Introduction

Badawi[1] introduced the concept of a 2-absorbing ideal in a commutative ring. A proper ideal I of a commutative ring R is said to be a 2-absorbing ideal, if whenever $a, b, c \in R$,

$abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. Payrovi and Babaei[2] extended the concept of 2-absorbing ideals in commutative rings.

Badawi[3] introduced 2-absorbing primary ideals in commutative rings. A proper ideal I of a commutative ring R is said to be a 2-absorbing primary ideal, if whenever $a, b, c \in R$, $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Mustafanasab and Darani[4] extended the concepts of 2-absorbing primary and weakly 2-absorbing primary ideals in commutative rings.

Manjarekar and Bingi[5] introduced 2-absorbing primary elements in multiplicative lattices. They defined, a proper element $q \in L$ to be a 2-absorbing primary if for every $a, b, c \in L$, $abc \leq q$ implies either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ca \leq \sqrt{q}$.

Celikel et al.[[6], [7]] introduced and studied ϕ-2-absorbing elements in multiplicative lattices. Let $\phi : L \to L \cup \{\emptyset\}$ be a function. A proper element q of L is said to be a ϕ-2-absorbing element of L if whenever $a, b, c \in L$ with $abc \leq q$ and $abc \nleq \phi(q)$ implies either $ab \leq q$ or $ac \leq q$ or $bc \leq q$. Celikel et al.[7] introduced ϕ-2-absorbing primary elements in multiplicative lattices as a generalization of ϕ-2-absorbing elements. Let $\phi : L \to L \cup \{\emptyset\}$ be a function. A proper element q of L is said to be a ϕ-2-absorbing primary element of L if whenever $a, b, c \in L$ with $abc \leq q$ and $abc \nleq \phi(q)$ implies either $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$.

Wasadikar and Gaikwad[8] introduced the concept of a 2-absorbing ideal in a lattice. A proper ideal I of a lattice L is said to be a 2-absorbing ideal, if whenever $a, b, c \in L$, $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

In this paper we introduce the concepts of the radical of an ideal (denoted by $\sqrt{I}$), the primary ideal and the 2-absorbing primary ideal in a lattice. It is shown that for an ideal I of a lattice L, $\sqrt{I}$ is a prime ideal of a lattice L if and only if $\sqrt{I}$ is a primary ideal of L. Similarly, it is shown that for an ideal I of a lattice L, $\sqrt{I}$ is a 2-absorbing ideal of a lattice L if and only if $\sqrt{I}$ is a 2-absorbing primary ideal of L. We prove that $I_1 \times L_2$ is a 2-absorbing primary ideal of $L = L_1 \times L_2$ if and only if I is a 2-absorbing primary ideal of L_1, where I_1 is a proper ideal of L_1.

The undefined terms are from Gratzer[9].

2. Preliminaries

We generalize the concepts of primary, 2-absorbing and 2-absorbing primary ideals from ring theory to lattices.

Definition 2.1

Let I be an ideal of a lattice L. We define the radical of I as the intersection of all prime ideals containing I and we denote it as $\sqrt{I}$.

Remark 2.1

If there does not exist a prime ideal containing an ideal I in a lattice L then $\sqrt{I} = L$.

Remark 2.2

In a distributive lattice L, an ideal I is the intersection of all prime ideals containing it see Gratzer [9, p. 75] i.e.$I = \sqrt{I}$.

However, this may or may not hold in a non distributive lattice.

Example 2.1

Consider the ideal $I = (n]$ of the lattice shown in Fig. 1. The lattice is non distributive. We observe that $I = \sqrt{I}$ since $\sqrt{I} = (p] \cap (r] = (n]$.

The following example shows that $I \subset \sqrt{I}$.

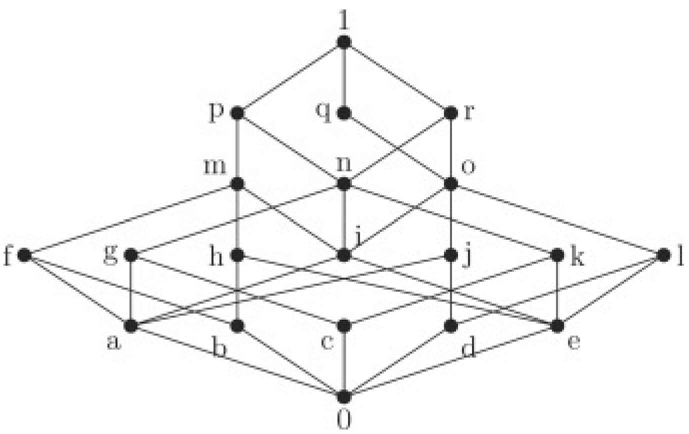

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Fig. 1.

Example 2.2

Consider the ideal $I = (l]$ of the lattice shown in Fig. 1. We observe that $I \subset \sqrt{I}$ since $\sqrt{I} = (q] \cap (r] = (o]$.

Definition 2.2

Let L be a lattice. A proper ideal I of L is called primary if $a, b \in L$ and $a \wedge b \in I$ imply that either $a \in I$ or $b \in \sqrt{I}$.

Example 2.3

In the lattice L shown in Fig. 1, consider the ideal $I = (m]$. Then $\sqrt{I} = (p]$. The ideal I is a primary ideal.

Consider the ideal $I = (j]$ of a lattice shown in Fig. 1. Then $\sqrt{I} = (q] \cap (r] = (o]$. The ideal I is not a primary ideal since $f \wedge g = a \in I$ but neither $f \in I$ nor $g \in I$. Also, neither $f \in \sqrt{I}$ nor $g \in \sqrt{I}$.

Definition 2.3

Let L be a lattice. A proper ideal I of L is called 2-absorbing if for $a, b, c \in L$, $a \wedge b \wedge c \in I$ then either $a \wedge b \in I$ or $a \wedge c \in I$ or $b \wedge c \in I$.

Example 2.4

Consider the lattice L shown in Fig. 1. Here the ideal $(n]$ is a 2-absorbing ideal.

Consider the ideal $I = (b]$ of a lattice shown in Fig. 1. The ideal I is not a 2-absorbing ideal since $i \wedge j \wedge l = 0 \in I$ but neither $i \wedge j = a \in I$ nor $i \wedge l = e \in I$ nor $j \wedge l = d \in I$.

Definition 2.4

Let L be a lattice. A proper ideal I of L is called a 2-absorbing primary if for $a, b, c \in L$, $a \wedge b \wedge c \in I$ then either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$.

Example 2.5

In the lattice shown in Fig. 1, the ideal $(f]$ is a 2-absorbing primary.

Consider the lattice of divisors of 30. Let $I = (0]$. Then $\sqrt{I} = (6] \cap (10] \cap (15] = (0]$. However, $6 \wedge 10 \wedge 15 = 0 \in I$, but neither $6 \wedge 10 = 2 \in I$ nor $6 \wedge 15 = 3 \in I$ nor $10 \wedge 15 = 5 \in I$. Hence I is not a 2-absorbing primary ideal.

3. Some properties of 2-absorbing primary ideals

The proofs ofLemma 3.1, Lemma 3.2, Lemma 3.3 are obvious.

Lemma 3.1

If I is a prime ideal of a lattice L, then I is a primary ideal of L.

Remark 3.1

The following example shows that the converse ofLemma 3.1 does not hold.

Example 3.1

Consider the lattice L shown in Fig. 1. For the ideal $I = (m]$, $\sqrt{I} = (p]$ as $(p]$ is the only prime ideal of L containing I. I is a primary ideal. However $k \wedge l = e \in I$ but neither $k \in I$ nor $l \in I$. Thus I is not a prime ideal.

Lemma 3.2

If I is a primary ideal of a lattice L, then I is a 2-absorbing primary ideal of L.

Remark 3.2

The following example shows that the converse ofLemma 3.2 does not hold.

Example 3.2

Consider the ideal $I = (l]$ of the lattice shown in Fig. 1. Thus $\sqrt{I} = (q] \cap (r] = (o]$. Here I is a 2-absorbing primary ideal. We note that $g \wedge h = 0 \in I$. However, neither $g \in I$ nor $h \in I$. Also, neither $g \in \sqrt{I}$ nor $h \in \sqrt{I}$. Hence I is not a primary ideal of L.

Lemma 3.3

If I is a 2-absorbing ideal of a lattice L, then I is a 2-absorbing primary ideal of L.

Remark 3.3

The following example shows that the converse ofLemma 3.3 does not hold.

Example 3.3

Consider the ideal $I = (0]$ of the lattice shown in Fig. 1. Thus $\sqrt{I} = (p] \cap (q] \cap (r] = (i]$. Here I is a 2-absorbing primary ideal. However $i \wedge j \wedge l = 0 \in I$, but neither $i \wedge j = a \in I$ nor $i \wedge l = e \in I$ nor $j \wedge l = d \in I$. Hence I is not a 2-absorbing ideal of L.

The following lemma is from Wasadikar and Gaikwad[8].

Lemma 3.4

Let P_1 and P_2 be two distinct prime ideals of a lattice L, then $P_1 \cap P_2$ is a 2-absorbing ideal of L.

Definition 3.1

Let I be an ideal of a lattice L. We define I as a P-primary ideal of L if P is the only prime ideal containing I.

Example 3.4

In the lattice shown in Fig. 1, the ideal $(m]$ is a P-primary ideal, where $P = (p]$ is the only prime ideal containing $(m]$.

The following result is an analog of[3, Theorem 2.4 (2)].

Theorem 3.1

Let L be a lattice. Suppose that I_1 is a P_1-primary ideal of L for some prime ideal P_1 of L and I_2 is a P_2-primary ideal of L for some prime ideal P_2 of L. Then $I_1 \cap I_2$ is a 2-absorbing primary ideal of L.

Proof

Let $I = I_1 \cap I_2$. Then $\sqrt{I} = P_1 \cap P_2$. Now suppose that $x \wedge y \wedge z \in I$ for some $x, y, z \in L$, $x \wedge z \notin \sqrt{I}$ and $y \wedge z \notin \sqrt{I}$. Then $x, y, z \notin \sqrt{I} = P_1 \cap P_2$. By Lemma 3.4, $\sqrt{I} = P_1 \cap P_2$ is a 2-absorbing ideal of L. Since $x \wedge z, y \wedge z \notin \sqrt{I}$, we have $x \wedge y \in \sqrt{I}$.

We show that $x \wedge y \in I$. Since $x \wedge y \in \sqrt{I} \subseteq P_1$, we may assume that $x \in P_1$. As $x \notin \sqrt{I}$ and $x \wedge y \in \sqrt{I} \subseteq P_2$, we conclude that $x \notin P_2$ and $y \in P_2$. Since $y \in P_2$ and $y \notin \sqrt{I}$, we have $y \notin P_1$.

If $x \in I_1$ and $y \in I_2$, then $x \wedge y \in I$ and we are done. We may assume that $x \notin I_1$. Since I_1 is a P_1-primary ideal and $x \notin I_1$, we have $y \wedge z \in P_1$. Since $y \in P_2$ and $y \wedge z \in P_1$, we have $y \wedge z \in \sqrt{I}$, which is a contradiction. Thus $x \in I_1$.

Since I_2 is a P_2-primary ideal of L and if $y \notin I_2$ then $x \wedge z \in P_2$. Since $x \in P_1$ and $x \wedge z \in P_1$, we have $x \wedge z \in \sqrt{I}$, which is a contradiction. Thus $y \in I_2$. Hence $x \wedge y \in I$. □

Theorem 3.2

Let I be a proper ideal of a lattice L such that $\sqrt{I}$ is a prime ideal of L. Then I is a 2-absorbing primary ideal of L.

Proof

Suppose that $a \wedge b \wedge c \in I$ for some $a, b, c \in L$ and $a \wedge b \notin I$.

(a) Suppose that $a \wedge b \notin \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal of L, $c \in \sqrt{I}$ and so $a \wedge c \in \sqrt{I}$ and $b \wedge c \in \sqrt{I}$.

(b) Suppose that $a \wedge b \in \sqrt{I}$. As $\sqrt{I}$ is a prime ideal, we have either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Hence $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$. Thus I is a 2-absorbing primary ideal of L. □

Remark 3.4

However, the converse ofTheorem 3.2 need not hold.

Example 3.5

Consider the ideal $I = (l]$ of the lattice shown in Fig. 1. Thus $\sqrt{I} = (q] \cap (r] = (o]$. Here I is a 2-absorbing primary ideal. However, $g \wedge h = 0 \in \sqrt{I}$, but neither $g \in \sqrt{I}$ nor $h \in \sqrt{I}$. Thus $\sqrt{I}$ is not a prime ideal of L.

Theorem 3.3

Let I be an ideal of a lattice L. Then $\sqrt{I}$ is a prime ideal of L if and only if $\sqrt{I}$ is a primary ideal of L.

Proof

Suppose that $\sqrt{I}$ is a prime ideal of L. If $a \wedge b \in \sqrt{I}$ then either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. As $\sqrt{I} = \sqrt{\sqrt{I}}$, either $a \in \sqrt{I}$ or $b \in \sqrt{\sqrt{I}}$. Hence $\sqrt{I}$ is a primary ideal of L.

Conversely, suppose that $\sqrt{I}$ is a primary ideal of L. Let $a \wedge b \in \sqrt{I}$. As $\sqrt{I}$ is a primary ideal, either $a \in \sqrt{I}$ or $b \in \sqrt{\sqrt{I}} = \sqrt{I}$. Thus $\sqrt{I}$ is a prime ideal of L. □

Similarly, we can prove the following characterization for 2-absorbing and 2-absorbing primary ideals of a lattice L.

Theorem 3.4

Let I be an ideal of a lattice L. Then $\sqrt{I}$ is a 2-absorbing ideal of L if and only if $\sqrt{I}$ is a 2-absorbing primary ideal of L.

Theorem 3.5

Let I be a 2-absorbing primary ideal of a lattice L and suppose that $x \wedge y \wedge J \subseteq I$ for some $x, y \in L$ and some ideal J of L. If $x \wedge y \notin I$, then $x \wedge J \subseteq \sqrt{I}$ or $y \wedge J \subseteq \sqrt{I}$.

Proof

Let $x \wedge y \notin I$. Suppose that $x \wedge J \nsubseteq \sqrt{I}$ and $y \wedge J \nsubseteq \sqrt{I}$. Then there exist some j_1 and some j_2 in J such that $x \wedge j_1 \notin \sqrt{I}$ and $y \wedge j_2 \notin \sqrt{I}$. As $x \wedge y \wedge j_1 \in I$, we have $y \wedge j_1 \in \sqrt{I}$ since I is a 2-absorbing primary ideal. Similarly, $x \wedge y \wedge j_2 \in I$ implies $x \wedge j_2 \in \sqrt{I}$.

Since $x \wedge y \wedge (j_1 \vee j_2) \in I$ and $x \wedge y \notin I$, we have either $x \wedge (j_1 \vee j_2) \in \sqrt{I}$ or $y \wedge (j_1 \vee j_2) \in \sqrt{I}$. Suppose that $x \wedge (j_1 \vee j_2) \in \sqrt{I}$. Therefore, $(x \wedge j_1) \vee (x \wedge j_2) \leq x \wedge (j_1 \vee j_2) \in \sqrt{I}$ and so $(x \wedge j_1) \vee (x \wedge j_2) \in \sqrt{I}$. Hence $x \wedge j_2 \in \sqrt{I}$ and $x \wedge j_1 \in \sqrt{I}$, which is a contradiction.

Similarly, if $y \wedge (j_1 \vee j_2) \in \sqrt{I}$ then $(y \wedge j_1) \vee (y \wedge j_2) \in \sqrt{I}$. Hence $y \wedge j_1 \in \sqrt{I}$ and $y \wedge j_2 \in \sqrt{I}$, which is a contradiction. Hence $x \wedge J \subseteq \sqrt{I}$ or $y \wedge J \subseteq \sqrt{I}$. □

Remark 3.5

The converse of Theorem 3.5 does not hold.

Example 3.6

Consider the ideal $I = (l]$ of the lattice shown in Fig. 1. Thus $\sqrt{I} = (o]$. I is a 2-absorbing primary ideal. Consider the ideal $J = (e]$. Now, $h \wedge i \wedge J = J \subseteq I$, $h \wedge J = J \subseteq I$ and $i \wedge J = J \subseteq I$, but $h \wedge i \in I$.

We give a characterization of a 2-absorbing primary ideal, which is an analog of[3, Theorem 2.19].

Theorem 3.6

Let I be a proper ideal of a lattice L. Then I is a 2-absorbing primary ideal if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq \sqrt{I}$ or $I_2 I_3 \subseteq \sqrt{I}$.

Proof

Let I be an ideal of L such that if $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$ or $I_1 I_3 \subseteq \sqrt{I}$ or $I_2 I_3 \subseteq \sqrt{I}$. We show that I is a 2-absorbing primary ideal of L. Let $a \wedge b \wedge c \in I$ for $a, b, c \in L$. This implies that $(a] \wedge (b] \wedge (c] \subseteq I$. Let $I_1 = (a]$, $I_2 = (b]$ and $I_3 = (c]$. By hypothesis, either $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq \sqrt{I}$ or $I_2 I_3 \subseteq \sqrt{I}$. Hence either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$. Thus I is a 2-absorbing primary ideal of L.

Conversely, suppose that I is a 2-absorbing primary ideal. Let $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L. Suppose that $I_1 I_2 \nsubseteq I$. We show that $I_1 I_3 \subseteq \sqrt{I}$ or $I_2 I_3 \subseteq \sqrt{I}$. Suppose that $I_1 I_3 \nsubseteq \sqrt{I}$ and $I_2 I_3 \nsubseteq \sqrt{I}$. Then there exist $q_1 \in I_1$ and $q_2 \in I_2$ such that $q_1 \wedge I_3 \nsubseteq \sqrt{I}$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$. As $q_1 \wedge q_2 \wedge I_3 \subseteq I$, we have $q_1 \wedge q_2 \in I$ by Theorem 3.5. Since $I_1 I_2 \nsubseteq I$, we have $a \wedge b \notin I$ for some $a \in I_1$, $b \in I_2$. Since $a \wedge b \wedge I_3 \subseteq I$ and $a \wedge b \notin I$, we have $a \wedge I_3 \subseteq \sqrt{I}$ or $b \wedge I_3 \subseteq \sqrt{I}$ by Theorem 3.5. We consider three cases.

Case 1: Suppose that $a \wedge I_3 \subseteq \sqrt{I}$ but $b \wedge I_3 \nsubseteq \sqrt{I}$. Since $q_1 \wedge b \wedge I_3 \subseteq I$ and $b \wedge I_3 \nsubseteq \sqrt{I}$ and $q_1 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $q_1 \wedge b \in I$ by Theorem 3.5. Since $(a \vee q_1) \wedge b \wedge I_3 \subseteq I$ and $a \wedge I_3 \subseteq \sqrt{I}$, but $q_1 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$. Since $b \wedge I_3 \nsubseteq \sqrt{I}$ and $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that

$(a \vee q_1) \wedge b \in I$ by Theorem 3.5. Since $(a \wedge b) \vee (q_1 \wedge b) \leq (a \vee q_1) \wedge b \in I$, we have $(a \wedge b) \vee (q_1 \wedge b) \in I$. Thus $q_1 \wedge b \in I$ and $a \wedge b \in I$, a contradiction.

Case 2: Suppose that $b \wedge I_3 \subseteq \sqrt{I}$, but $a \wedge I_3 \nsubseteq \sqrt{I}$.

Since $a \wedge q_2 \wedge I_3 \subseteq I$ and $a \wedge I_3 \nsubseteq \sqrt{I}$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $a \wedge q_2 \in I$ by Theorem 3.5. Since $a \wedge (b \vee q_2) \wedge I_3 \subseteq I$ and $b \wedge I_3 \subseteq \sqrt{I}$, but $q_2 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$. Since $a \wedge I_3 \nsubseteq \sqrt{I}$ and $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $a \wedge (b \vee q_2) \in I$ by Theorem 3.5. Since $(a \wedge b) \vee (a \wedge q_2) \leq a \wedge (b \vee q_2) \in I$, we have $(a \wedge b) \vee (a \wedge q_2) \in I$. Thus $a \wedge q_2 \in I$ and $a \wedge b \in I$, a contradiction.

Case 3: $a \wedge I_3 \subseteq \sqrt{I}$ and $b \wedge I_3 \subseteq \sqrt{I}$.

Since $b \wedge I_3 \subseteq \sqrt{I}$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$. Since $q_1 \wedge (b \vee q_2) \wedge I_3 \subseteq I$ and $q_1 \wedge I_3 \nsubseteq \sqrt{I}$ and $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $q_1 \wedge (b \vee q_2) \in I$ by Theorem 3.5. As $(q_1 \wedge b) \vee (q_1 \wedge q_2) \leq q_1 \wedge (b \vee q_2) \in I$, we have $(q_1 \wedge b) \vee (q_1 \wedge q_2) \in I$. Hence $b \wedge q_1 \in I$. Since $a \wedge I_3 \subseteq \sqrt{I}$ and $q_1 \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$. Since $(a \vee q_1) \wedge q_2 \wedge I_3 \subseteq I$ and $q_2 \wedge I_3 \nsubseteq \sqrt{I}$ and $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge q_2 \in I$ by Theorem 3.5. As $(a \wedge q_2) \vee (q_1 \wedge q_2) \leq (a \vee q_1) \wedge q_2 \in I$, we have $(a \wedge q_2) \vee (q_1 \wedge q_2) \in I$. Hence $a \wedge q_2 \in I$. Now, since $(a \vee q_1) \wedge (b \vee q_2) \wedge I_3 \subseteq I$ and $(a \vee q_1) \wedge I_3 \nsubseteq \sqrt{I}$ and $(b \vee q_2) \wedge I_3 \nsubseteq \sqrt{I}$, we conclude that $(a \vee q_1) \wedge (b \vee q_2) \in I$ by Theorem 3.5. We conclude that $a \wedge b \in I$, a contradiction. Hence $I_1 I_3 \subseteq \sqrt{I}$ or $I_2 I_3 \subseteq \sqrt{I}$. □

Theorem 3.7

Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) *If P' is a prime ideal of L', then $f^{-1}(P')$ is a prime ideal of L.*

(2) *If f is an isomorphism and P is a prime ideal of L, then $f(P)$ is a prime ideal of L'.*

Proof

(1) Let $a \wedge b \in f^{-1}(P')$ for $a, b \in L$. Then $f(a \wedge b) \in P'$. Hence $f(a) \wedge f(b) \in P'$. This implies that either $f(a) \in P'$ or $f(b) \in P'$. That is either $a \in f^{-1}(P')$ or $b \in f^{-1}(P')$. Thus $f^{-1}(P')$ is a prime ideal of L.

(2) Let $a' \wedge b' \in f(P)$ for $a', b' \in L'$. Then there exist some $a, b \in L$ such that $f(a) = a'$ and $f(b) = b'$. Thus $f(a) \wedge f(b) = a' \wedge b' \in f(P)$. Thus $f(a \wedge b) \in f(P)$. Hence $a \wedge b \in P$. As P is a prime ideal of L, either $a \in P$ or $b \in P$. That is either $f^{-1}(a') \in P$ or $f^{-1}(b') \in P$. Hence either $a' \in f(P)$ or $b \in f(P)$. Thus $f(P)$ is a prime ideal of L'.

□

Theorem 3.8

Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) *If I' is an ideal of L', then* $f^{-1}\left(\sqrt{I'}\right) = \sqrt{f^{-1}(I')}$.

(2) *If f is an isomorphism and I is an ideal of L, then* $f\left(\sqrt{I}\right) = \sqrt{f(I)}$.

Proof

(1) Let P_i''s be all prime ideals of L' containing I' where $i \in \Lambda$. Then $f^{-1}\left(\sqrt{I'}\right) = f^{-1}(\bigcap P_i')$. Which implies that $f^{-1}\left(\sqrt{I'}\right) = \bigcap f^{-1}(P_i')$. As P_i''s are prime ideals of L', $f^{-1}(P_i')$'s are prime ideals of L, by Theorem 3.7 (1) and as $I' \subseteq \bigcap P_i'$, we have $f^{-1}(I') \subseteq \bigcap f^{-1}(P_i')$. Which implies that $\bigcap f^{-1}(P_i') = \sqrt{f^{-1}(I')}$. Hence $f^{-1}\left(\sqrt{I'}\right) = \sqrt{f^{-1}(I')}$.

(2) Let P_i's be all prime ideals of L containing I where $i \in \Lambda$. Then $f\left(\sqrt{I}\right) = f(\bigcap P_i)$. This implies that $f\left(\sqrt{I}\right) = \bigcap f(P_i)$. As P_i's are prime ideals of L, $f(P_i)$'s are prime ideals of L', by Theorem 3.7 (2) and as $I \subseteq \bigcap P_i$, we have $f(I) \subseteq \bigcap f(P_i)$. Implies that $\bigcap f(P_i) = \sqrt{f(I)}$. Hence $f\left(\sqrt{I}\right) = \sqrt{f(I)}$. □

The following result is an analog of[3, Theorem 2.20].

Theorem 3.9

Let $f : L \to L'$ be a homomorphism of lattices. Then the following statements hold:

(1) *If I' is a 2-absorbing primary ideal of L', then $f^{-1}(I')$ is a 2-absorbing primary ideal of L.*

(2) *If f is an isomorphism and I is a 2-absorbing primary ideal of L, then $f(I)$ is a 2-absorbing primary ideal of L'.*

Proof

(1) Let $a, b, c \in L$ such that $a \wedge b \wedge c \in f^{-1}(I')$. Then $f(a \wedge b \wedge c) = f(a) \wedge f(b) \wedge f(c) \in I'$. As I' is 2-absorbing primary ideal, we have either $f(a) \wedge f(b) \in I'$ or $f(a) \wedge f(c) \in \sqrt{I'}$ or $f(b) \wedge f(c) \in \sqrt{I'}$. That is either $a \wedge b \in f^{-1}(I')$ or $a \wedge c \in f^{-1}\left(\sqrt{I'}\right)$ or $b \wedge c \in f^{-1}\left(\sqrt{I'}\right)$. As

$f^{-1}\left(\sqrt{I'}\right)=\sqrt{f^{-1}\left(I'\right)}$, by Theorem 3.8 (1), $a\wedge b\in f^{-1}\left(I'\right)$ or $a\wedge c\in\sqrt{f^{-1}\left(I'\right)}$ or $b\wedge c\in\sqrt{f^{-1}\left(I'\right)}$. Thus $f^{-1}\left(I'\right)$ is a 2-absorbing primary ideal of L.

(2) Let $a',b',c'\in L'$ and $a'\wedge b'\wedge c'\in f(I)$. Then there exist $a,b,c\in L$ such that $f(a)=a'$, $f(b)=b'$, $f(c)=c'$ and $f(a)\wedge f(b)\wedge f(c)=a'\wedge b'\wedge c'\in f(I)$. That is $f(a)\wedge f(b)\wedge f(c)\in f(I)$. Hence $a\wedge b\wedge c\in I$. As I is a 2-absorbing primary ideal, we have either $a\wedge b\in I$ or $a\wedge c\in\sqrt{I}$ or $b\wedge c\in\sqrt{I}$. That is either $f^{-1}\left(a'\wedge b'\right)\in I$ or $f^{-1}\left(a'\wedge c'\right)\in\sqrt{I}$ or $f^{-1}\left(b'\wedge c'\right)\in I$. Thus either $a'\wedge b'\in f(I)$ or $a'\wedge c'\in f\left(\sqrt{I}\right)$ or $b'\wedge c'\in f\left(\sqrt{I}\right)$. As $f\left(\sqrt{I}\right)=\sqrt{f(I)}$, by Theorem 3.8 (2) $a'\wedge b'\in f(I)$ or $a'\wedge c'\in\sqrt{f(I)}$ or $b'\wedge c'\in\sqrt{f(I)}$. Hence $f(I)$ is a 2-absorbing ideal of L'. □

4. 2-absorbing primary ideals in product lattices

In this section we prove some results on 2-absorbing primary ideals in product lattices. The notion of the product lattice is from Gratzer[9, p. 27].

The proof of the following theorem is obvious.

Theorem 4.1

Let $L=L_1\times L_2$, where L_1 and L_2 are lattices. Let P_i's and Q_j's be ideals of L_1 and L_2 respectively, where $i\in\Lambda_1$ and $j\in\Lambda_2$. Then $\bigcap\left(P_i\times Q_j\right)=\bigcap P_i\times\bigcap Q_j$.

Theorem 4.2

Let $L=L_1\times L_2$, where each L_i, $(i=1,2)$ is a lattice with 1. Then the following hold:

(1) *If I_1 is an ideal of L_1, then $\sqrt{I_1\times L_2}=\sqrt{I_1}.\times L_2$*

(2) *If I_2 is an ideal of L_2, then $\sqrt{L_1\times I_2}=L_1.\times\sqrt{I_2}$*

Proof

(1) Let $(a,b)\in\sqrt{I_1\times L_2}$. Thus $(a,b)\in\bigcap_{i\in\Lambda}\left(P_i\times L_2\right)$, where and P_i's are all prime ideals of a lattice L_1 containing I_1. Thus $a\in\bigcap_{i\in\Lambda}P_i$, $b\in L_2$. Thus $a\in\sqrt{I_1}$, $b\in L_2$ and so $(a,b)\in\sqrt{I_1}\times L_2$.

If $(a,b)\in\sqrt{I_1}\times L_2$ then $a\in\sqrt{I_1}$, $b\in L_2$. Thus $a\in\bigcap_{i\in\Lambda}P_i$, $b\in L_2$ and so $(a,b)\in\bigcap_{i\in\Lambda}\left(P_i\times L_2\right)$. i.e.$(a,b)\in\sqrt{I_1\times L_2}$. Hence $\sqrt{I_1\times L_2}=\sqrt{I_1}\times L_2$.

(2) Proof is similar to that of (1). □

The following characterization gives a relation between a 2-absorbing primary ideal of a product of two lattices and a 2-absorbing primary ideal of one of the lattice in this product.

Theorem 4.3

Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let I be a proper ideal of L_1. Then $I \times L_2$ is a 2-absorbing primary ideal if and only if I is a 2-absorbing primary ideal of L_1.

Proof

Suppose that $I \times L_2$ is a 2-absorbing ideal of L. Let $a \wedge b \wedge c \in I$ for $a, b, c \in L_1$. Then $(a \wedge b \wedge c, x) \in I \times L_2$ for $x \in L_2$. As $I \times L_2$ is a 2-absorbing primary ideal of L, either $(a \wedge b, x) \in I \times L_2$ or $(a \wedge c, x) \in \sqrt{I \times L_2}$ or $(b \wedge c, x) \in \sqrt{I \times L_2}$. Then either $(a \wedge b, x) \in I \times L_2$ or $(a \wedge c, x) \in \sqrt{I} \times L_2$ or $(b \wedge c, x) \in \sqrt{I} \times L_2$, by Theorem 4.2. Hence either $a \wedge b \in I$ or $a \wedge c \in \sqrt{I}$ or $b \wedge c \in \sqrt{I}$.

Conversely, suppose that I is a 2-absorbing primary ideal of L_1. Let $(a \wedge b \wedge c, x) \in I$ for $a, b, c \in L_1$ and $x \in L_2$. As I is a 2-absorbing primary ideal of L_1, either $(a \wedge b, x) \in I \times L_2$ or $(a \wedge c, x) \in \sqrt{I} \times L_2$ or $(b \wedge c, x) \in \sqrt{I} \times L_2$. That is either $(a \wedge b, x) \in I \times L_2$ or $(a \wedge c, x) \in \sqrt{I \times L_2}$ or $(b \wedge c, x) \in \sqrt{I \times L_2}$, by Theorem 4.2. □

Theorem 4.4

Let $L = L_1 \times L_2$, where L_1 and L_2 are lattices. Let I_1 and I_2 be proper ideals of L_1 and L_2 respectively. If $I = I_1 \times I_2$ is a 2-absorbing primary ideal of L then I_1 and I_2 are 2-absorbing primary ideals of L_1 and L_2 respectively.

Proof

Let $a \wedge b \wedge c \in I_1$ for some $a, b, c \in L_1$. Then $(a \wedge b \wedge c, x) \in I_1 \times I_2$ for $x \in I_2$. As $I_1 \times I_2$ is a 2-absorbing primary ideal, either $(a \wedge b, x) \in I_1 \times I_2$ or $(a \wedge c, x) \in \sqrt{I_1 \times I_2}$ or $(b \wedge c, x) \in \sqrt{I_1 \times I_2}$, that is either $(a \wedge b, x) \in I_1 \times I_2$ or $(a \wedge c, x) \in \sqrt{I_1} \times \sqrt{I_2}$ or $(b \wedge c, x) \in \sqrt{I_1} \times \sqrt{I_2}$, by Theorem 4.2. Hence $a \wedge b \in I_1$ or $a \wedge c \in \sqrt{I_1}$ or $b \wedge c \in \sqrt{I_1}$. Thus I_1 is a 2-absorbing primary ideal of L_1. Similarly, we can show that I_2 is a 2-absorbing primary ideal of L_2. □

Remark 4.1

The converse ofTheorem 4.4 need not hold.

Example 4.1

Consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as shown in Fig. 2. Consider the ideals $I_1 = \{0\}, I_2 = \{0\}$ of the lattices L_1 and L_2 respectively. Thus $I_1 \times I_2 = \{(0, 0)\}$ and

$\sqrt{I_1 \times I_2} = \{(0,0)\}$. The ideals I_1 and I_2 are 2-absorbing primary ideals of L_1 and L_2 respectively. But for $(a,1) \wedge (1,0) \wedge (b,1) = (0,0) \in I_1 \times I_2$, neither $(a,1) \wedge (1,0) = (a,0) \in I_1 \times I_2$ nor $(a,1) \wedge (b,1) = (0,1) \in I_1 \times I_2$ nor $(1,0) \wedge (b,1) = (b,0) \in I_1 \times I_2$. Thus $I_1 \times I_2$ is not a 2-absorbing primary ideal of L.

Now we give a characterization of a 2-absorbing primary ideal in a product of two lattices, which is an analog of[3, Theorem 2.23].

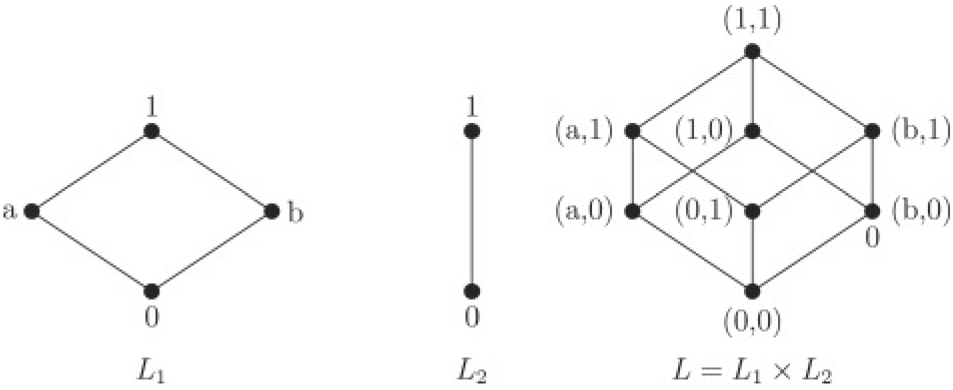

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Fig. 2.

Theorem 4.5

Let $L = L_1 \times L_2$, where L_1 and L_2 are bounded lattices. Let J be a proper ideal of L. Then the following statements are equivalent:

(1) *J is a 2-absorbing primary ideal of L.*

(2) *Either $J = I_1 \times L_2$ for some 2-absorbing primary ideal I_1 of L_1 or $J = L_1 \times I_2$ for some 2-absorbing primary ideal I_2 of L_2 or $J = I_1 \times I_2$ for some primary ideal I_1 of L_1 and some primary ideal I_2 of L_2.*

Proof

$(1) \Longrightarrow (2)$. Suppose that J is a 2-absorbing primary ideal of L. Then $J = I_1 \times I_2$ for some ideal I_1 of L_1 and some ideal I_2 of L_2.

Case 1: If $I_2 = L_2$ then $I_1 \neq L_1$. Thus $J = I_1 \times L_2$. Let $a \wedge b \wedge c \in I_1$ for some $a, b, c \in L_1$. Then $(a \wedge b \wedge c, x \wedge y \wedge z) \in I_1 \times L_2$, where $x, y, z \in L_2$. As J is a 2-absorbing primary ideal, we have either $(a \wedge b, x \wedge y) \in I_1 \times L_2$ or $(a \wedge c, x \wedge z) \in \sqrt{I_1 \times L_2}$ or $(b \wedge c, y \wedge z) \in \sqrt{I_1 \times L_2}$. By Lemma 3.1, either

$(a \wedge b, x \wedge y) \in I_1 \times L_2$ or $(a \wedge c, x \wedge z) \in \sqrt{I_1} \times L_2$ or $(b \wedge c, y \wedge z) \in \sqrt{I_1} \times L_2$. Thus I_1 is a 2-absorbing primary ideal of L_1.

Case 2: If $I_1 = L_1$ then $I_2 \neq L_2$. Thus $J = L_1 \times I_2$. Similarly, as in previous case, I_2 is a 2-absorbing primary ideal of L_2.

Case 3: Now if $I_1 \neq L_1$ and $I_2 \neq L_2$ then $J = I_1 \times I_2$. That is $\sqrt{J} = \sqrt{I_1} \times \sqrt{I_2}$. On the contrary, suppose that I_1 is not a primary ideal of L_1. Then there are $a, b \in L_1$ such that $a \wedge b \in I_1$ but neither $a \in I_1$ nor $b \in \sqrt{I_1}$. Let $x = (a, 1), y = (1, 0)$ and $c = (b, 1)$. Then $x \wedge y \wedge c = (a \wedge b, 0) \in J$ but neither $x \wedge y = (a, 0) \in J$ nor $x \wedge c = (a \wedge b, 1) \in \sqrt{J}$ nor $y \wedge c = (b, 0) \in \sqrt{J}$, which is a contradiction. Thus I_1 is a primary ideal of L_1. Suppose that I_2 is not a primary ideal of L_2. Then there exist $d, e \in L_2$ such that $d \wedge e \in I_2$ but neither $d \in I_2$ nor $e \in \sqrt{I_2}$. Let $x = (1, d), y = (0, 1)$ and $c = (1, e)$. Then $x \wedge y \wedge c = (0, d \wedge e) \in J$ but neither $x \wedge y = (0, d) \in J$ nor $x \wedge c = (1, d \wedge e) \in \sqrt{J}$ nor $y \wedge c = (0, e) \in \sqrt{J}$, which is a contradiction. Thus I_2 is a primary ideal of L_2.

$(2) \implies (1)$. Suppose that $J = I_1 \times L_2$ for some 2-absorbing primary ideal I_1 of L_1. Let $(a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3) \in I_1 \times L_2$. Then $a_1 \wedge a_2 \wedge a_3 \in I_1$. As I_1 is 2-absorbing primary ideal of L_1, we have either $a_1 \wedge a_2 \in I_1$ or $a_1 \wedge a_3 \in \sqrt{I_1}$ or $a_2 \wedge a_3 \in \sqrt{I_1}$. That is either $(a_1, b_1) \wedge (a_2, b_2) \in I_1 \times L_2$ or $(a_1, b_1) \wedge (a_3, b_3) \in \sqrt{I_1} \times L_2$ or $(a_2, b_2) \wedge (a_3, b_3) \in \sqrt{I_1} \times L_2$. Hence either $(a_1, b_1) \wedge (a_2, b_2) \in I_1 \times L_2$ or $(a_1, b_1) \wedge (a_3, b_3) \in \sqrt{I_1 \times L_2}$ or $(a_2, b_2) \wedge (a_3, b_3) \in \sqrt{I_1 \times L_2}$ by Theorem 4.2. Thus $J = I_1 \times L_2$ is a 2-absorbing primary ideal of L. Similarly $L_1 \times I_2$ is a 2-absorbing primary ideal of L. Suppose that $J = I_1 \times I_2$ for some primary ideal I_1 of L_1 and some primary ideal I_2 of L_2. Then $P = I_1 \times L_2$ and $Q = L_1 \times I_2$ are primary ideals of L. Hence $P \cap Q = I_1 \times I_2$. Thus $J = I_1 \times I_2$ is a 2-absorbing primary ideal, by Theorem 3.1. □

The following theorem is a generalization ofTheorem 4.5, which is an analog of[3, Theorem 2.24].

Theorem 4.6

Let $L = L_1 \times L_2 \cdots \times L_n$, where $2 \leq n < \infty$, and $L_1, L_2, \ldots, L_n$ are lattices. Let J be a proper ideal of L. Then the following statements are equivalent.

(1) *J is a 2-absorbing primary ideal of L.*

(2) *Either $J = \prod_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, \ldots, n\}$, I_k is a 2-absorbing primary ideal of L_k, and $I_t = L_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $J = \prod_{t=1}^{n} I_t$ such that for some $k, m \in \{1, 2, \ldots, n\}$, I_k is a primary ideal of L_k, I_m is a primary ideal of L_m, and $I_t \neq L_t$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k, m\}$.*

Proof

$(1) \Leftrightarrow (2)$ We prove this theorem by induction on n. Assume $n = 2$. Then by Theorem 4.5, the result holds. Thus suppose that $3 \leq n < \infty$ and assume that the result is valid when $K = L_1 \times L_2 \cdots L_{n-1}$. Now we prove the result when $L = K \times L_n$. By Theorem 4.5, J is a 2-absorbing primary ideal of L if and only if either $J = A \times L_n$ for some 2-absorbing primary ideal A of K or $J = K \times A_n$ for some 2-absorbing primary ideal A_n of L_n or $J = A \times A_n$ for some primary ideal A of K and some primary ideal A_n of L_n. Now observe that a proper ideal B of K is a primary ideal of K if and only if $B = \prod_{t=1}^{n-1} I_t$ such that for some $k \in \{1, 2, \ldots, n-1\}$, I_k is a primary ideal of L_k, and $I_t \neq L_t$ for every $t \in \{1, 2, \ldots, n-1\} \setminus \{k, m\}$. □

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