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Existence and Uniqueness of Solutions of Fractional Differential Equations with Deviating Arguments under Integral Boundary Conditions

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Abstract. The aim of this paper is to develop a monotone iterative technique by introducing upper and lower solutions to Riemann-Liouville fractional differential equations with deviating arguments and integral boundary conditions. As an application of this technique, existence and uniqueness results are obtained.

1. Introduction

Differential equations with deviating arguments arise in various branches of science, engineering, economics and so on (see [4, 8] and the references therein). Many researchers have studied the existence, uniqueness, continuous dependence, and stability of solutions of nonlinear fractional differential equations (see [1, 2, 3, 5, 6, 7, 10, 11, 12, 15, 16, 17, 19, 20, 25, 30, 33, 34]). The monotone iterative technique [23] combined with the method of upper and lower solutions provides an effective mechanism to prove constructive existence results for nonlinear differential equations. The monotone technique is an interesting and powerful tool to deal with existence results for fractional differential equations. In 2008, the monotone technique for fractional differential equations with initial conditions was first developed by Lakshmikantham and Vatsala [25]. Later, a series of papers appeared in the literature to prove existence and uniqueness of solution of various problems with initial conditions, boundary conditions, integral boundary conditions, nonlinear boundary conditions, and periodic boundary conditions for fractional differential equations, (see, for ex-

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ample [13, 14, 18, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 35, 37, 38, 39, 40, 41, 42, 43] and the references therein). However, work on fractional differential equations with deviating argument is rare. In this paper, we study the following problem for the Riemann-Liouville fractional differential equation with a deviating argument and integral boundary conditions:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f\left(t, u(t), u(\theta(t))\right), \ t \in J = [0,T], \\ u(0) = \lambda \int_0^T u(s)ds + d, \quad d \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f \in C\left(J \times \mathbb{R}^2, \mathbb{R}\right)$, $\theta \in C\left(J, J\right)$, $\theta(t) \leq t$, $t \in J$, $\lambda \geq 0$, $0 < \alpha < 1$. The paper is organized as follows. In Section 2, we introduce some useful definitions and basic lemmas. In Section 3, we study the uniqueness of a solution for the problem (1.1) using the Banach fixed point theorem. In Section 4, we develop the monotone method and apply it to obtain existence and uniqueness results for Riemann-Liouville fractional differential equations with deviating arguments and integral boundary conditions.

2. Preliminaries

For the reader's convenience, we present some necessary definitions and lemmas from the theory of fractional calculus. In addition, we prove some basic results which are useful for further discussion.

Definition 2.1.([21, 36]) For $\alpha > 0$, the integral

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

is called the *Riemann-Liouville fractional integral* of order α.

Definition 2.2.([21, 36]) The Riemann-Liouville derivative of order α ($n-1 < \alpha \leq n$) can be written as

$$D_{0^+}^{\alpha} u(t) = \left(\frac{d}{dt}\right)^n \left(I_{0^+}^{n-\alpha} u(t)\right) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \ t > 0.$$

Lemma 2.1.([21]) *Let $u \in C^n[0,T]$, $\alpha \in (n-1, n)$, $n \in \mathbb{N}$. Then for $t \in J$,*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0).$$

Consider the space $C_{1-\alpha}(J, \mathbb{R}) = \left\{u \in C\left((0,T], \mathbb{R}\right) : t^{1-\alpha} u \in C\left(J, \mathbb{R}\right)\right\}$.

Lemma 2.2.([9]) *Let $m \in C_{1-\alpha}(J, \mathbb{R})$ where for some $t_1 \in (0,T]$,*

$$m(t_1) = 0 \text{ and } m(t) \leq 0 \text{ for } 0 \leq t \leq t_1.$$

Then it follows that

$$D^{\alpha}m(t_1) \geq 0.$$

Lemma 2.3. *Let $f \in C\left(J \times \mathbb{R}^2, \mathbb{R}\right)$. A function $u \in C_{1-\alpha}(J, \mathbb{R})$ is a solution of the problem (1.1) if and only if u is a solution of the integral equation*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u(\theta(s))\right) ds + \lambda \int_0^T u(s) ds + d. \tag{2.1}$$

Proof. Assume that u satisfies the problem (1.1). From the first equation of the problem (1.1) and Lemma 2.1, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u(\theta(s))\right) ds + \lambda \int_0^T u(s) ds + d.$$

Conversely, assume that $u \in C_{1-\alpha}(J, \mathbb{R})$ satisfies the integral equation (2.1). Applying the Riemann-Liouville operator D^{α}_{0+} to both sides of the integral equation (2.1), we have

$$\begin{aligned} D^{\alpha}_{0+} u(t) &= D^{\alpha}_{0+} \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u(\theta(s))\right) ds + \lambda \int_0^T u(s) ds + d \right) \\ D^{\alpha}_{0+} u(t) &= f\left(t, u(t), u(\theta(t))\right). \end{aligned}$$

In addition, we have $u(0) = \lambda \int_0^T u(s) ds + d$ from the integral equation (2.1). The proof is complete. □

Lemma 2.4. *Suppose that $\{u_\epsilon\}$ is a family of continuous functions defined on J, for each $\epsilon > 0$, which satisfies*

$$\begin{cases} D^{\alpha}_{0+} u_\epsilon(t) = f\left(t, u_\epsilon(t), u_\epsilon(\theta(t))\right), \\ u_\epsilon(0) = \lambda \int_0^T u_\epsilon(s) ds + d, \end{cases} \tag{2.2}$$

where $\left|f\left(t, u_\epsilon(t), u_\epsilon(\theta(t))\right)\right| \leq M$ for $t \in J$. Then the family $\{u_\epsilon\}$ is equicontinuous on J.

Proof. For $0 \leq t_1 < t_2 \leq T$, consider

$$\begin{aligned} |u_\epsilon(t_1) - u_\epsilon(t_2)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1-s)^{\alpha-1} f\left(s, u_\epsilon(s), u_\epsilon(\theta(s))\right) ds - \right. \\ &\qquad \left. \int_0^{t_2} (t_2-s)^{\alpha-1} f\left(s, u_\epsilon(s), u_\epsilon(\theta(s))\right) ds \right| \\ &\leq \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right) \end{aligned}$$

$$\leq \frac{M}{\Gamma(\alpha+1)}\left[t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha\right]$$
$$\leq \frac{2M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha < \epsilon,$$

provided that $|t_2 - t_1| < \delta = \left[\frac{\epsilon\Gamma(\alpha+1)}{2M}\right]^{\frac{1}{\alpha}}$, proving the result. □

3. Uniqueness of Solution

In this section, we obtain the uniqueness of solution of the problem (1.1) for Riemann-Liouville fractional differential equations with deviating argument and integral boundary conditions.

Theorem 3.1. *Assume that*

(i) $f \in C\left(J \times \mathbb{R}^2, \mathbb{R}\right),\ \theta(t) \in C\left(J, J\right),\ \theta \leq t,\ t \in J,$

(ii) *there exists nonnegative constants M and N such that function f satisfies*

$$|f(t,u_1,u_2) - f(t,\upsilon_1,\upsilon_2)| \leq M\,|u_1-\upsilon_1| + N\,|u_2-\upsilon_2|\,,$$

for all $t \in J$, $u_i, \upsilon_i \in \mathbb{R}$, $i = 1, 2$. If $\lambda < \frac{\Gamma(\alpha+1)-T^\alpha(M+N)}{T\Gamma(\alpha+1)}$, then the problem (1.1) has a unique solution.

Proof. Consider the operator T defined by

$$\left(Tu\right)(t) = \lambda\int_0^T u(s)ds + d + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, u(s), u(\theta(s))\right)ds.$$

Now, we show that $T : C_{1-\alpha}(J,\mathbb{R}) \to C_{1-\alpha}(J,\mathbb{R})$ is a contraction operator. For any $u, \upsilon \in C_{1-\alpha}(J,\mathbb{R})$, we have

$$\begin{aligned}
\|Tu - T\upsilon\|_C &= \max_{t\in J}|(Tu)(t) - (T\upsilon)(t)| \\
&\leq \max_{t\in J}\lambda\int_0^T |u(s)-\upsilon(s)|\,ds + \max_{t\in J}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left|f\left(s,u(s),u(\theta(s))\right) - f\left(s,\upsilon(s),\upsilon(\theta(s))\right)\right| ds \\
&\leq \lambda\int_0^T ds\,\|u-\upsilon\|_C + \max_{t\in J}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left[|M\left(u(s)-\upsilon(s)\right)| + |N\left(u(\theta(s))-\upsilon(\theta(s))\right)|\right] ds \\
&\leq \lambda T\,\|u-\upsilon\|_C + \max_{t\in J}\frac{(M+N)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\|u-\upsilon\|_C\,ds \\
&\leq \lambda T\,\|u-\upsilon\|_C + \max_{t\in J}\frac{(M+N)t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\eta)^{\alpha-1}d\eta\,\|u-\upsilon\|_C \\
&\leq \left[\lambda T + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(M+N\right)\right]\|u-\upsilon\|_C\,.
\end{aligned}$$

Therefore, $\|Tu - Tv\|_C < \|u - v\|_C$. By the Banach fixed point theorem, the operator T has a unique fixed point, i.e. the problem (1.1) has a unique solution. The proof is complete. □

Corollary 3.1. *Let M, N be constants, $\sigma \in C_{1-\alpha}(J, \mathbb{R})$. The linear problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \ 0 < \alpha < 1, \ t \in J, \\ u(0) = \lambda \int_0^T u(s)ds + d, \ d \in \mathbb{R}, \end{cases} \tag{3.1}$$

has a unique solution.

Proof. It follows from the Theorem 3.1. □

4. Monotone Iterative Method

In this section, we prove the existence and uniqueness of solution for the problem (1.1) by monotone iterative technique combined with the method of upper and lower solutions. Now we define the functional interval as follows:

$$[v_0, w_0] = \{u \in C_{1-\alpha}(J, \mathbb{R}) : v_0(t) \le u(t) \le w_0(t) \ \forall t \in J\}.$$

First, we prove the following comparison result, which plays an important role in our further discussion.

Lemma 4.1. *Let $\theta \in C(J, J)$ where $\theta(t) \le t$ on J. Suppose that $p \in C_{1-\alpha}(J, \mathbb{R})$ satisfies the inequalities*

$$\begin{cases} D_{0+}^{\alpha} p(t) \le -Mp(t) - Np(\theta(t)) \equiv Fp(t), \quad t \in J, \\ \quad p(0) \le 0, \end{cases} \tag{4.1}$$

where M and N are constants. If

$$-(1 + T^{\alpha})[M + N] < \Gamma(1 + \alpha), \tag{4.2}$$

then $p(t) \le 0$ for all $t \in J$.

Proof. Consider $p_\epsilon(t) = p(t) - \epsilon(1 + t^\alpha)$, $\epsilon > 0$. Then

$$\begin{aligned} D_{0+}^{\alpha} p_\epsilon(t) &= D_{0+}^{\alpha} p(t) - D_{0+}^{\alpha} \epsilon(1 + t^\alpha) \\ &\le Fp(t) - \frac{\epsilon}{t^\alpha \Gamma(1-\alpha)} - \epsilon\Gamma(1 + \alpha) \\ &= Fp_\epsilon(t) + \epsilon\left[-M(1 + t^\alpha) - N(1 + t^\alpha) - \frac{1}{t^\alpha\Gamma(1-\alpha)} - \Gamma(1 + \alpha)\right] \\ &< Fp_\epsilon(t) + \epsilon\left[-(1 + t^\alpha)(M + N) - \Gamma(1 + \alpha)\right] < Fp_\epsilon(t) \end{aligned}$$

and

$$p_\epsilon(0) = p(0) - \epsilon(1 + t^\alpha) < 0.$$

We prove that $p_\epsilon(t) < 0$ *on* J. Assume that$p_\epsilon(t) \nless 0$ on J. Thus there exists a $t_1 \in (0,T]$ such that $p_\epsilon(t_1) = 0$ and $p_\epsilon(t) < 0, t \in (0,t_1)$. In view of Lemma 2.2, we have $D^\alpha_{0+} p_\epsilon(t_1) \geq 0$. It follows that

$$0 < Fp_\epsilon(t) = -Np_\epsilon(\theta(t_1)).$$

If $N = 0$, then $0 < 0$, which is a contradiction. If $-N > 0$, then $p_\epsilon(\theta(t_1)) > 0$, which is again a contradiction. This proves that $p_\epsilon(t) < 0$ on J. So $p(t) - \epsilon(1 + t^\alpha) < 0$ on J. Taking $\epsilon \to 0$, we obtain required result. □

Definition 4.1. A pair of functions $[\upsilon_0,\ w_0]$ in $C_{1-\alpha}(J,\mathbb{R})$ are called *lower and upper solutions* of the problem (1.1) if

$$D^\alpha_{0+}\upsilon_0(t) \leq f\left(t, \upsilon_0(t), \upsilon_0(\theta(t))\right), \quad \upsilon_0(0) \leq \textstyle\int_0^T \upsilon_0(s)ds + d \tag{4.3}$$

and

$$D^\alpha_{0+}w_0(t) \geq f\left(t, w_0(t), w_0(\theta(t))\right), \quad w_0(0) \geq \textstyle\int_0^T w_0(s)ds + d. \tag{4.4}$$

Theorem 4.1. *Assume that*

(i) $f \in C\left(J \times \mathbb{R}^2, \mathbb{R}\right),\ \theta \in C\left(J, J\right),\ \theta(t) \leq t,\ t \in J,$

(ii) *functions υ_0 and w_0 in $C_{1-\alpha}(J,\mathbb{R})$ are lower and upper solutions of the problem (1.1) such that $\upsilon_0(t) \leq w_0(t)$ on J,*

(iii) *there exists nonnegative constants M, N such that function f satisfies the condition*

$$f(t, u_1, u_2) - f(t, \upsilon_1, \upsilon_2) \geq -M(u_1 - \upsilon_1) - N(u_2 - \upsilon_2), \tag{4.5}$$

for $\upsilon_0(t) \leq \upsilon_1 \leq u_1 \leq w_0(t)$, $\upsilon_0(\theta(t)) \leq \upsilon_2 \leq u_2 \leq w_0(\theta(t))$.

Then there exists monotone sequences $\{\upsilon_n(t)\}$ and $\{w_n(t)\}$ in $C_{1-\alpha}(J,\mathbb{R})$ such that

$$\{\upsilon_n(t)\} \to \upsilon(t) \text{ and } \{w_n(t)\} \to w(t) \text{ as } n \to \infty$$

for all $t \in J$, where υ and w are minimal and maximal solutions of the problem (1.1) respectively and $\upsilon(t) \leq u(t) \leq w(t)$ on J.

Proof. For any $\eta \in C_{1-\alpha}(J,\mathbb{R})$ such that $\eta \in [\upsilon_0, w_0]$, we consider the following linear problem:

$$\begin{cases} D^\alpha_{0+}u(t) = f\left(t, \eta(t), \eta(\theta(t))\right) + M\left[\eta(t) - u(t)\right] + N\left[\eta(\theta(t)) - u(\theta(t))\right], \\ \qquad u(0) = \int_0^T u(s)ds + d, \end{cases} \tag{4.6}$$

By Corollary 3.1, the linear problem (4.6) has a unique solution $u(t)$.

Next, we define the iterates as follows and construct the sequences $\{\upsilon_n\}$, $\{u_n\}$

$$\begin{cases} D_{0^+}^{\alpha}\upsilon_{n+1}(t) = f\left(t, \upsilon_n(t), \upsilon_n(\theta(t))\right) - \\ \qquad M\left[\upsilon_{n+1}(t) - \upsilon_n(t)\right] - N\left[\upsilon_{n+1}(\theta(t)) - \upsilon_n(\theta(t))\right], \\ \upsilon_{n+1}(0) = \int_0^T \upsilon_n(s)ds + d, \end{cases} \tag{4.7}$$

and

$$\begin{cases} D_{0^+}^{\alpha}w_{n+1}(t) = f\left(t, w_n(t), w_n(\theta(t))\right) - \\ \qquad M\left[w_{n+1}(t) - w_n(t)\right] - N\left[w_{n+1}(\theta(t)) - w_n(\theta(t))\right], \\ w_{n+1}(0) = \int_0^T w_n(s)ds + d, \end{cases} \tag{4.8}$$

Clearly, the existence of solutions υ_{n+1} and w_{n+1} of the problems (4.7) and (4.8), respectively, follows from the above arguments. Further, by setting $n = 0$ in the problems (4.7), (4.8), we get the existence of solutions υ_1 and w_1, respectively. We show that $\upsilon_0(t) \le \upsilon_1(t) \le w_1(t) \le w_0(t)$. Set $p(t) = \upsilon_1(t) - \upsilon_0(t)$. Since υ_0 is the lower solution of the problem (4.7), we have

$$\begin{aligned} D_{0^+}^{\alpha}p(t) &= D_{0^+}^{\alpha}\upsilon_1(t) - D_{0^+}^{\alpha}\upsilon_0(t) \\ &\ge f\left(t, \upsilon_0(t), \upsilon_0(\theta(t))\right) - f\left(t, \upsilon_0(t), \upsilon_0(\theta(t))\right) - \\ &\quad M\left[\upsilon_1(t) - \upsilon_0(t)\right] - N\left[\upsilon_1(\theta(t)) - \upsilon_0(\theta(t))\right] \\ &\ge -Mp(t) - Np(\theta(t)) \end{aligned}$$

and

$$p(0) = \upsilon_1(0) - \upsilon_0(0) \ge \int_0^T \upsilon_0(s)ds + d - \int_0^T \upsilon_0(s)ds - d = 0.$$

From Lemma 4.1, we obtain $p(t) \ge 0$, which implies that $\upsilon_1(t) \ge \upsilon_0(t)$ on J. Similarly, we can prove $\upsilon_1(t) \le w_1(t)$ and $w_1(t) \le w_0(t)$ on J. Thus $\upsilon_0(t) \le \upsilon_1(t) \le w_1(t) \le w_0(t)$. Assume that for some $k > 1$,

$$\upsilon_{k-1}(t) \le \upsilon_k(t) \le w_k(t) \le w_{k-1}(t) \text{on } J.$$

We claim that $\upsilon_k(t) \le \upsilon_{k+1}(t) \le w_{k+1}(t) \le w_k(t)$ on J. To prove our claim, set $p(t) = \upsilon_{k+1}(t) - \upsilon_k(t)$. Then we have

$$\begin{aligned} D_{0^+}^{\alpha}p(t) &= D_{0^+}^{\alpha}\upsilon_{k+1}(t) - D_{0^+}^{\alpha}\upsilon_k(t) \\ &= f\left(t, \upsilon_k(t), \upsilon_k(\theta(t))\right) - M\left[\upsilon_{k+1}(t) - \upsilon_k(t)\right] - \\ &\quad N\left[\upsilon_{k+1}(\theta(t)) - \upsilon_k(\theta(t))\right] - f\left(t, \upsilon_{k-1}(t), \upsilon_{k-1}(\theta(t))\right) + \\ &\quad M\left[\upsilon_k(t) - \upsilon_{k-1}(t)\right] + N\left[\upsilon_k(\theta(t)) - \upsilon_{k-1}(\theta(t))\right] \\ &\ge -M\left[\upsilon_{k+1}(t) - \upsilon_k(t)\right] - N\left[\upsilon_{k+1}(\theta(t)) - \upsilon_k(\theta(t))\right] \\ &\ge -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$\begin{aligned} p(0) &= \upsilon_{k+1}(0) - \upsilon_k(0) = \int_0^T \upsilon_k(s)ds + d - \int_0^T \upsilon_{k-1}(s)ds - d \\ &\ge \int_0^T \left[\upsilon_k(s) - \upsilon_k(s)\right] ds = 0. \end{aligned}$$

By Lemma 4.1, we obtain $p(t) \geq 0$, implying that $\upsilon_{k+1}(t) \geq \upsilon_k(t)$ for all $t \in J$. Similarly, we can prove $\upsilon_{k+1}(t) \leq w_{k+1}(t)$ and $w_{k+1}(t) \leq w_k(t)$ for all $t \in J$. From the principle of mathematical induction, we have

$$\upsilon_0 \leq \upsilon_1 \leq \upsilon_1 \leq ... \leq \upsilon_k \leq w_k \leq ... \leq w_2 \leq w_1 \leq w_0 \text{ on } J. \tag{4.9}$$

Clearly, the sequences $\{\upsilon_n\}, \{w_n\}$ are monotonic and uniformly bounded. Further we observe that $\{D^{\alpha}_{0+}\upsilon_n\}$ and $\{D^{\alpha}_{0+}w_n\}$ are also uniformly bounded on J, in view of the relations (4.7), (4.8). Applying Lemma 2.4 we can conclude that sequences $\{\upsilon_n\}, \{w_n\}$ are equicontinuous. Hence by the Ascoli-Arzela theorem the sequences $\{_n\}, \{w_n\}$ converge uniformly to υ and w on J respectively.

Now, we prove that υ and w are the minimal and maximal solutions of the problem (1.1). Let u be any solution of the problem (1.1) different from υ and w. So there exists a k such that $\upsilon_k(t) \leq u(t) \leq w_k(t)$ on J. Set $p(t) = u(t) - \upsilon_{k+1}(t)$. Then we have

$$\begin{aligned} D^{\alpha}_{0+}p(t) &= D^{\alpha}_{0+}u(t) - D^{\alpha}_{0+}\upsilon_{k+1}(t) \\ &= f\left(t, u(t), u(\theta(t))\right) - f\left(t, \upsilon_k(t), \upsilon_k(\theta(t))\right) + \\ &\quad M\left[\upsilon_{k+1}(t) - \upsilon_k(t)\right] + N\left[\upsilon_{k+1}(\theta(t)) - \upsilon_k(\theta(t))\right] \\ &\geq -M\left[u(t) - \upsilon_{k+1}(t)\right] - N\left[u(\theta(t)) - \upsilon_{k+1}(\theta(t))\right] \\ &\geq -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = u(0) - \upsilon_{k+1}(0) = \int_0^T \left[u(s) - \upsilon_k(s)\right] ds \geq 0.$$

By Lemma 4.1, we obtain $p(t) \geq 0$, implying that $u(t) \geq \upsilon_{k+1}(t)$ for all k on J. Similarly, we can prove $u(t) \leq w_{k+1}(t)$ for all k on J. Since $\upsilon_0(t) \leq u(t) \leq u_0(t)$ on J. By induction it follows that $\upsilon_k(t) \leq u(t)$ and $u(t) \leq w_k(t)$ for all k. Thus $\upsilon_k(t) \leq u(t) \leq w_k(t)$ on J. Taking the limit as $k \to \infty$, we obtain $\upsilon(t) \leq u(t) \leq w(t)$ on J. Thus the functions $\upsilon(t), w(t)$ are the minimal and maximal solutions of the problem (1.1). The proof is complete. □

Next we prove the uniqueness of solution of the problem (1.1) as follows.

Theorem 4.2. *Assume that*

(i) *all the conditions of the Theorem 4.1 hold,*

(ii) *there exists nonnegative constants M, N such that the function f satisfies the condition*

$$f(t, u_1, u_2) - f(t, \upsilon_1, \upsilon_2) \leq M(u_1 - \upsilon_1) + N(u_2 - \upsilon_2), \tag{4.10}$$

for $\upsilon_0(t) \leq \upsilon_1 \leq u_1 \leq w_0(t)$, $\upsilon_0(\theta(t)) \leq \upsilon_2 \leq u_2 \leq w_0(\theta(t))$.

Then the problem (1.1) has a unique solution.

Proof. We know $\upsilon(t) \leq w(t)$ on J. It is sufficient to prove that $\upsilon(t) \geq w(t)$ on J. Consider $p(t) = w(t) - \upsilon(t)$. Then we have

$$\begin{aligned} D_{0^+}^{\alpha} p(t) &= D_{0^+}^{\alpha} w(t) - D_{0^+}^{\alpha} \upsilon(t) \\ &= f\left(t, w(t), w(\theta(t))\right) - f\left(t, \upsilon(t), \upsilon(\theta(t))\right) \\ &\leq -M\left[\upsilon(t) - w(t)\right] - N\left[\upsilon(\theta(t)) - w(\theta(t))\right] \\ &= -Mp(t) - Np(\theta(t)) \end{aligned}$$

and

$$p(0) = w(0) - \upsilon(0) = \int_0^T \left[w(s) - \upsilon(s)\right] ds \leq 0.$$

By Lemma 4.1, we know $p(t) \leq 0$, implying that $\upsilon(t) \geq w(t)$, and the result follows. □

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